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Published in:
IEEE Transactions on Automatic Control

DOI:
10.1109/TAC.2024.3430416

Published: 01/01/2025

Document Version
Final published version

Region-Free Explicit Model-Predictive Control for Linear Systems on Hilbert Spaces

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***Abstract*—In this article, we extend discrete-time explicit model-predictive control (MPC) rigorously to linear distributed parameter systems. After formulating an MPC framework and giving a relevant Karush–Kuhn–Tucker theorem, we realize fast regionless explicit MPC by using the dual-active-set method QPKWIK. A Timoshenko beam with input and state constraints is used to demonstrate the efficacy of the design at controlling a continuous-time hyperbolic partial differential equation with constraints, using a discrete-time explicit MPC controller.**

***Index Terms*—Distributed parameter systems, optimal control, predictive control for linear systems.**

I. Introduction

Model-predictive control (MPC, also called *receding horizon control*) is a technique for approximately solving an infinite-horizon optimal control problem by instead solving a sequence of finite-horizon problems. This gives a near-optimal control signal implicitly, formed by the initial parts of the solutions of a sequence of optimization problems. In [1] and some other papers of the same time, such as [2] and [3], it was realized that these optimization problems can sometimes be solved explicitly, foreseeing major speedups compared to traditional MPC using naive online optimization. These seminal papers sparked intensive research on what is now called *explicit MPC*. Explicit MPC classically consists of two phases: the *offline phase*, where the piecewise affine optimal control step is precalculated and stored in advance via the process of *state-space exploration* (see the next paragraph), and the *online* phase, where at every time step, *point location* is performed in order to determine and retrieve the optimal control step.

Automatic state-space exploration, where the state space of the plant is divided into polygonal regions, on which the optimal control signal is piecewise affine, is tricky already in the finite-dimensional setting, due to degeneracy issues (see [4]). Algorithms that work around degeneracy have been proposed, e.g., in [5] and [6]. The complexity of the explicit control increases rapidly with both the state-space dimension of the plant and the prediction horizon, leading to challenges with storing the explicit control in a data structure that facilitates retrieving the optimal control [7]. Kvasnica et al. [8] have studied complexity reduction in explicit MPC.

Manuscript received 14 January 2024; revised 23 May 2024; accepted 11 July 2024. Date of publication 18 July 2024; date of current version 31 December 2024. The work of Jukka-Pekka Humaloja was supported in part by the Jenny and Antti Wihuri Foundation and in part by the Vilho, Yrjö and Kalle Väisälä Foundation. Recommended by Associate Editor V. Andrieu. *(Corresponding author: Mikael Kurula.)*

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Digital Object Identifier 10.1109/TAC.2024.3430416

Later, a much lighter explicit MPC technique appeared in [9], the so-called *region-free* approach; see also [8, Sec. 2.2] and the references therein. The region-free approach avoids state exploration and storage of the full piecewise affine controller, hence eliminating the offline phase altogether. Instead, in the online phase, it calculates the optimal control step on the fly, using the set of constraints active at the optimal control step associated with the present plant state. Such methods are referred to as *active set methods*, and prime examples are the dual-active-set method by Goldfarb and Idnani [10] and its refined version QPKWIK [11]. Recent work of Mönnigmann et al. [12], [13], [14] investigates the dynamics of the optimal active set and the explicit MPC controller, as functions of the horizon length.

While *explicit* MPC is introduced here as being new for infinite-dimensional systems, generic MPC theory for infinite-dimensional discrete-time (mainly nonlinear) plants has been developed, e.g., by Grüne and Pannek [15]. Altmüller [16] demonstrates that some of the central ideas in [15] are effective also for hyperbolic partial differential equations (PDEs). In this technical note, we concentrate on how to compute the MPC input explicitly and efficiently, for affinely constrained linear systems with quadratic cost, rather than on fundamental properties of MPC itself, such as stabilization or feasibility.

We contribute a simple, fast, light, and clean setup for *explicit* MPC of distributed parameter systems, which avoids all degeneracy issues that clutter the present literature. The proposed methodology is applied to a Timoshenko beam model with input and state constraints, where we show how one can control the continuous-time plant with the discretely generated control signal. While we focus on infinite-dimensional systems, we point out that this article provides a powerful implementation for the finite-dimensional literature as well. A preliminary version of the proposed approach was presented without proofs in [17].

MPC for infinite-dimensional continuous-time plants, usually without constraints, has been studied before; see, in particular, [18], but also, e.g., [19], [20], [21], and [22]. The continuous-time setting in these studies makes the analysis rather technical, and hoping to make an impact in engineering, we will prefer a more practically oriented and elementary approach in discrete time, which allows us to handle a finite number of general affine constraints, similar to [23] and [24].

The rest of this article is organized as follows. In Section II, we describe a general and flexible MPC setup, which we further reformulate as a standard parametric quadratic program (pQP) in Section III. In Section IV, we solve the pQP explicitly and make a connection to QPKWIK, thus completing the implementation of reliable explicit MPC for linear PDE systems. Finally, Section V concludes this article, where the setup is demonstrated on the Timoshenko beam.

II. Discrete-Time MPC Formulation

In this section, we describe a rather flexible MPC formulation: We allow time-dependent weights, time-dependent constraints, and a cost on the rate of change of control. We note that, similar to [1, Sec. 6], the basic formulation can easily be extended in several ways, in order to account, e.g., for reference tracking, measured disturbances, and soft constraints, but we omit those here for the sake of brevity.

Let U and X be *real* Hilbert spaces and let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(U, X)$ be bounded linear operators. Consider the discrete-time plant dynamics

$$x_{n+1} = Ax_n + Bu_n, \quad n \in \mathbb{N}_0 \tag{1}$$

where $u_n \in U$ is the *input* at the discrete time step n and $x_n \in X$ is the *state* at time n. The objective is to steer x_n in (1) to zero in such a way that a quadratic cost functional of the following form is minimized:

$$\sum_{n=0}^{\infty} \left(\left\langle \begin{bmatrix} Q & M \\ M^* & R \end{bmatrix} \begin{bmatrix} x_n \\ u_n \end{bmatrix}, \begin{bmatrix} x_n \\ u_n \end{bmatrix} \right\rangle + \langle V(u_n - u_{n-1}), u_n - u_{n-1} \rangle \right) \tag{2}$$

for some appropriate weights Q, M, R, and V and some arbitrarily fixed $u_{-1} \in U$, say $u_{-1} = 0$, while satisfying some affine constraints on the input u and the state x, which we denote by $\begin{bmatrix} x_n \\ u_{n-1} \\ u_n \end{bmatrix} \in \mathcal{W}$ for $n = 0, 1, \ldots$. In the optimal control problem (1) and (2), the initial state x_0 is given, and the optimization variable is the control signal $\{u_n\}_{n=0}^{\infty}$. The optimization problem (3) and (4) below should be interpreted analogously.

One may try to solve the above optimal control problem approximately by approximating the cost (2) by a finite sum. Choosing a *horizon* N and denoting $\mathbf{u}' := (u'_k)_{k=0}^{N-1}$, the control step u_n at time n is chosen as the first element $u_{*,0}$ of the minimizer $\mathbf{u}_* = (u_{*,k})_{k=0}^{N-1}$ of the cost functional

$$J\left(\mathbf{u}', \begin{bmatrix} x_n \\ u_{n-1} \end{bmatrix}\right) := \langle Px'_N, x'_N \rangle + \langle V_N u'_{N-1}, u'_{N-1} \rangle + \sum_{k=0}^{N-1} \left(\left\langle \begin{bmatrix} Q_k & M_k \\ M_k^* & R_k \end{bmatrix} \begin{bmatrix} x'_k \\ u'_k \end{bmatrix}, \begin{bmatrix} x'_k \\ u'_k \end{bmatrix} \right\rangle + \langle V_k(u'_k - u'_{k-1}), u'_k - u'_{k-1} \rangle \right) \tag{3}$$

where the terminal penalty $P = P^*$ and the weights $Q_k = Q_k^* \in \mathcal{L}(X)$, $R_k = R_k^*$, $V_k = V_k^* \in \mathcal{L}(U)$, and $M_k \in \mathcal{L}(U, X)$ satisfy $P, V_N, V_k, \begin{bmatrix} Q_k & M_k \\ M_k^* & R_k \end{bmatrix} \geq 0$ for all $k = 0, \ldots, N-1$. The minimization of the cost functional (3) is subject to constraints of the form

$$\begin{cases} x'_{k+1} = A'x'_k + B'u'_k, \\ x'_0 = x_n, \\ u'_{-1} = u_{n-1}, \end{cases} \qquad \begin{bmatrix} x'_N \\ u'_{N-1} \end{bmatrix} \in \mathcal{T}$$

$$\begin{bmatrix} x'_k \\ u'_k \\ u'_{k-1} \end{bmatrix} \in \mathcal{W}_k, \quad 0 \leq k \leq N-1 \tag{4}$$

where $\mathcal{W}_k$ and $\mathcal{T}$ represent some stage and terminal constraints introduced in (5) and (6), respectively, and ideally, the *prediction model* is perfect, i.e., $(A', B') = (A, B)$. The obtained control step u_n is applied to the plant, and at the next time step, a new optimization is carried out. We *assume* that the MPC procedure is stabilizing and feasible, and refer to [15, Sec. 5–7] for an introduction to these topics.

By the above procedure, MPC solves a constrained quadratic optimization problem at every time step n in order to compute the control action u_n. *The point of explicit MPC* is to express the optimal input sequence $\mathbf{u}'$ of (3) and (4) subject to the constraints *explicitly* as a piecewise affine function of the parameter $\theta_n := \begin{bmatrix} x_n \\ u_{n-1} \end{bmatrix}$, rather than as the implicit minimizer of the constrained optimization problem described earlier.

In the following, we assume that the stage constraints in (4) can be written in the affine form, for all $k = 0, 1, \ldots, N-1$,

$$\begin{bmatrix} x'_k \\ u'_{k-1} \\ u'_k \end{bmatrix} \in \mathcal{W}_k \iff d_k - \mathcal{E}_k x'_k - \mathcal{F}_k u'_{k-1} - E_k u'_k \in [0, \infty)^{p_k} \tag{5}$$

and that the terminal constraint can be written as

$$\begin{bmatrix} x'_N \\ u'_{N-1} \end{bmatrix} \in \mathcal{T} \iff \widehat{d} - \widehat{E}x'_N - \widehat{F}u'_{N-1} \in [0, \infty)^{\hat{p}} \tag{6}$$

with $d_k \in [0, \infty)^{p_k}$, $\widehat{d} \in [0, \infty)^{\hat{p}}$, $\mathcal{E}_k \in \mathcal{L}(X; \mathbb{R}^{p_k})$, $\mathcal{F}_k, E_k \in \mathcal{L}(U; \mathbb{R}^{p_k})$, $\widehat{F} \in \mathcal{L}(U; \mathbb{R}^{\hat{p}})$, and $\widehat{E} \in \mathcal{L}(X; \mathbb{R}^{\hat{p}})$ all given as part of the control problem, where $\hat{p}$ and p_k for all $k = 0 \ldots, N-1$ are nonnegative integers. Unlike our work here, in the literature, it is common to assume that the constraints on the state are decoupled from the constraints on the control. However, this is restrictive in practice, e.g., in the situation where one already has a low-level controller in closed loop with the plant that one wants to control by MPC, and one needs to avoid saturating the input of the low-level controller.

Remark 1: If there is no penalty on the rate of change of control, i.e., $V_k = 0$ for all $k \in \mathbb{N}_0$ in (2), then one can take $\mathcal{F}_k = 0$ in (5) and $\widehat{F} = 0$ in (6). Moreover, there may then be no need to keep track of u'_{k-1}, so that at time step n, the constraints reduce to $\begin{bmatrix} x'_k \\ u'_k \end{bmatrix} \in \mathcal{W}_k$ for $k = 0, 1, \ldots, N-1, x_N \in \mathcal{T}$, and the parameter reduces to $\theta_n = x_n$. In this case, there is no need to specify a u_{-1} when initializing the system at time $n = 0$.

III. Writing the Optimization Step as a pQP

Iterating the dynamics $x'_{k+1} = Ax'_k + Bu'_k$ of the prediction model and using $x'_0 = x_n$, we get the solution formula

$$x'_k = A^k x_n + \sum_{j=0}^{k-1} A^j B u'_{k-1-j}, \quad k \geq 0$$

and using this, we can derive bounded operators $\widetilde{A}$ and $\widetilde{B}$ with

$$\mathbf{x}' := (x'_k)_{k=1}^{N} = \widetilde{A}x_n + \widetilde{B}\mathbf{u}'.$$

Note the difference in the indexing: in the vector $\mathbf{x}'$, the predicted states are at time steps $1, \ldots, N$ and the predicted control moves in $\mathbf{u}'$ are at time steps $0, \ldots, N-1$.

With $\widetilde{Q}_P := Q_1 \oplus \cdots \oplus Q_{N-1} \oplus P \geq 0$, $\widetilde{R} := R_0 \oplus \cdots \oplus R_{N-1} \geq 0$, $\widetilde{M}_0 := \begin{bmatrix} M_0 & 0 & \ldots & 0 \end{bmatrix}$,

$$\widetilde{M} := \begin{bmatrix} 0 & M_1 & 0 & \ldots & 0 \\ 0 & 0 & M_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & M_{N-1} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad \widetilde{V}_0 := \begin{bmatrix} -V_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\widetilde{V} := \begin{bmatrix} V_0+V_1 & -V_1 & 0 & \dots & 0 \\ -V_1 & V_1+V_2 & -V_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \ddots & \ddots & -V_{N-1} \\ 0 & \dots & 0 & -V_{N-1} & V_{N-1}+V_N \end{bmatrix}$$

and denoting $\theta_n := \begin{bmatrix} x_n \\ u_{n-1} \end{bmatrix}$, the cost functional (3) can be written as

$$J(\mathbf{u}',\theta_n) = \left\langle \begin{bmatrix} Q_0 + \widetilde{A}^*\widetilde{Q}_P\widetilde{A} & 0 \\ 0 & V_0 \end{bmatrix} \theta_n, \theta_n \right\rangle + \langle H\mathbf{u}',\mathbf{u}'\rangle + 2\langle \mathbf{u}', F\theta_n\rangle \quad (7)$$

$$H := \widetilde{B}^*\widetilde{Q}_P\widetilde{B} + \widetilde{R} + \widetilde{V} + \widetilde{B}^*\widetilde{M} + \widetilde{M}^*\widetilde{B} \quad \text{and}$$

$$F := \begin{bmatrix} \widetilde{B}^*\widetilde{Q}_P\widetilde{A} + \widetilde{M}^*\widetilde{A} + \widetilde{M}_0^* & \widetilde{V}_0 \end{bmatrix}. \quad (8)$$

Since we want to minimize $J(\mathbf{u}',\theta_n)$ by varying $\mathbf{u}'$, for a fixed given $\theta_n = \begin{bmatrix} x_n \\ u_{n-1} \end{bmatrix}$, the first term in (7) does not influence the location of the minimum, and it can be omitted in the optimization.

The particular case when H is *coercive* is important, i.e., when $H^* = H$ and

$$\langle H\mathbf{u}',\mathbf{u}'\rangle \geq \varepsilon\|\mathbf{u}'\|^2, \quad \mathbf{u}' \in U^N \quad (9)$$

for some $\varepsilon > 0$ independent of $\mathbf{u}'$. This is guaranteed, e.g., if $\widetilde{V}$ is coercive (see Lemma 2); indeed, observe that

$$H - \widetilde{V} := \widetilde{B}^*\widetilde{Q}_P\widetilde{B} + \widetilde{R} + \widetilde{B}^*\widetilde{M} + \widetilde{M}^*\widetilde{B} \geq 0$$

since $\langle (H-\widetilde{V})\mathbf{u}',\mathbf{u}'\rangle$ equals $J(\mathbf{u}',0)$ in (7), or equivalently in (3), with $V_k = 0$ for all k, and we assumed $\begin{bmatrix} Q_k & M_k \\ M_k^* & R_k \end{bmatrix}$, $P \geq 0$. Moreover, H inherits coercivity from $\widetilde{R}$ in case $M_k = 0$ for all $k = 1,\dots,N-1$.

Lemma 2: The operator $\widetilde{V}$ is positive semidefinite. If V_k are coercive for all (possibly apart from one) $k \in \{0,1,\dots,N\}$, then $\widetilde{V}$ is coercive.

Proof: For an arbitrary $\mathbf{u}' \in U^N$, we have

$$\langle \widetilde{V}\mathbf{u}',\mathbf{u}'\rangle = \langle V_0u_0',u_0'\rangle + \langle V_Nu_{N-1}',u_{N-1}'\rangle + \sum_{k=1}^{N-1}\langle V_k(u_k'-u_{k-1}'), u_k'-u_{k-1}'\rangle.$$

The right-hand side is clearly nonnegative as $\langle V_ku,u\rangle \geq 0$ for all $u \in U$ and $k \in \{0,1,\dots,N\}$, see the paragraph after (4), and this implies that $\widetilde{V}$ is positive semidefinite.

Let us now assume that every V_k is coercive, possibly apart from one, i.e., that there exists $\varepsilon_k \geq 0$, all strictly positive, apart from at most one, such that $\langle V_ku,u\rangle \geq \varepsilon_k\|u\|^2$ independently of $u \in U$ for all $k \in \{0,1,\dots,N\}$. Using this and the reverse triangle inequality, we get

$$\langle \widetilde{V}\mathbf{u}',\mathbf{u}'\rangle \geq \varepsilon_0\|u_0'\|^2 + \varepsilon_N\|u_{N-1}'\|^2 + \sum_{k=1}^{N-1}\varepsilon_k\left(\|u_k'\| - \|u_{k-1}'\|\right)^2 = \langle \widetilde{\varepsilon}\mathbf{u},\mathbf{u}\rangle_{\mathbb{R}^N} \quad (10)$$

where we denote $\mathbf{u} := (\|u_k'\|)_{k=0}^{N-1} \in \mathbb{R}^N$ and $\widetilde{\varepsilon}$ is defined as $\widetilde{V}$ with V_k replaced by ε_k; hence, $\widetilde{\varepsilon}$ is an $N \times N$ matrix, rather than an operator on U^N.

The right-hand side in (10) is nonnegative, and it can only be zero if $\|u_k\| = 0$ for all $k \in \{0,1,\dots,N-1\}$, i.e., $\mathbf{u} = 0$, and this implies that $\widetilde{\varepsilon} > 0$. Consequently, we have $\langle \widetilde{\varepsilon}\mathbf{u},\mathbf{u}\rangle \geq \lambda_{\min}(\widetilde{\varepsilon})\|\mathbf{u}\|^2$, where $\lambda_{\min}(\widetilde{\varepsilon}) > 0$ denotes the smallest eigenvalue of $\widetilde{\varepsilon}$. Finally, noting that $\|\mathbf{u}\|^2 = \|\mathbf{u}'\|^2$, we have that $\widetilde{V}$ is coercive, because

$$\langle \widetilde{V}\mathbf{u}',\mathbf{u}'\rangle \geq \langle \widetilde{\varepsilon}\mathbf{u},\mathbf{u}\rangle \geq \lambda_{\min}(\widetilde{\varepsilon})\,\|\mathbf{u}'\|^2. \qquad \square$$

Consequently, one can set $V_0 = 0$ to avoid having to specify u_{-1} when initializing the system at time $n = 0$, in case u_{k-1}' does not appear in the stage constraints (5).

Completing the square in (7), one gets that

$$f(\mathbf{z}) := \frac{1}{2}\langle H\mathbf{z},\mathbf{z}\rangle \quad (11)$$

satisfies the following for all $\theta_n = \begin{bmatrix} x_n \\ u_{n-1} \end{bmatrix}$:

$$f(\mathbf{z}) = \frac{1}{2}J(\mathbf{u}',\theta_n) + g(\theta_n), \quad \text{as long as} \quad \mathbf{z} = \mathbf{u}' + H^{-1}F\theta_n.$$

Moreover, the inverse H^{-1} is bounded with norm at most $1/\varepsilon$ because of (9). Hence, we will focus on minimizing $f(\mathbf{z})$.

Letting $\widetilde{p} < \infty$ denote the total number of constraints in (5) and (6), one can, in a similar way to the above, derive a vector $W \in \mathbb{R}^{\widetilde{p}}$ and bounded linear operators $S \in \mathcal{L}(X \times U, \mathbb{R}^{\widetilde{p}})$ (or $S \in \mathcal{L}(X, \mathbb{R}^{\widetilde{p}})$) and $G \in \mathcal{L}(U^N, \mathbb{R}^{\widetilde{p}})$, such that all these constraints can be written compactly as

$$W + S\theta_n - G\mathbf{z} \geq 0 \quad (12)$$

where the inequality is understood componentwise in $\mathbb{R}^{\widetilde{p}}$.

IV. Explicit Solution of the pQP

We will now solve explicitly the minimization of f in (11) subject to the constraints (12), i.e., the following pQP, which is to be solved by the MPC controller at every time step

$$\operatorname{argmin}_{\mathbf{z} \in U^N} \frac{1}{2}\langle H\mathbf{z},\mathbf{z}\rangle \quad \text{s.t.} \quad W + S\theta_n - G\mathbf{z} \geq 0 \quad (13)$$

where $W \in \mathbb{R}^{\widetilde{p}}$ and H, S, and G are all bounded operators.

We start with a simple result on existence and uniqueness, which is contained in [25, Prop. 2.3.3, Th. 3.3.4].

Lemma 3: Assume that the optimization problem (13) is *feasible*, i.e., that the *feasible set* $\mathcal{S}(\theta_n)$, which consists of all $\mathbf{z}$ satisfying the constraint in (13), is nonempty. Moreover, assume that the operators G and H are bounded and that H is *coercive*. Then, (13) has a unique solution $\mathbf{z}_*$.

We next work toward a Hilbert space extension of the Karush–Kuhn–Tucker optimality conditions used in the explicit MPC literature. The first step is taken in the following lemma, whose second part is also contained in [25, Th. 9.6.1].

Lemma 4: Let Z be a real Hilbert space, and let f and $f_k, 1 \leq k \leq \widetilde{p}$, be a finite number of convex functionals on Z. Moreover, assume that f and all f_k are Gâteaux differentiable on Z. Then, statement 2) implies statement 1).

1) The point $z_* \in Z$ minimizes $f(z)$ subject to the constraints $f_k(z) \geq 0$ for all $k = 1,\dots,\widetilde{p}$.
2) The point $z_* \in Z$ satisfies $f_k(z_*) \geq 0$ for all $k = 1,\dots,\widetilde{p}$, and there are $\lambda_k \geq 0$, $k = 1,\dots,\widetilde{p}$, such that:
   a) $\lambda_k f_k(z_*) = 0$ for all $k = 1,\dots,\widetilde{p}$;
   b) the Gâteaux derivatives of f and f_k satisfy

$$f'(z_*) = \sum_{k=1}^{\widetilde{p}} \lambda_k f_k'(z_*).$$

Assume that the constraints satisfy the following Slater condition: There exists $z \in Z$, such that $f_k(z) > 0$ for all $k = 1, \dots, \widetilde{p}$. Then, the implication from 1) to 2) also holds.

Proof: First, note that all Hilbert spaces are locally convex. The implication from 2) to 1) is then contained in [26, Cor. 47.14, statement (4)], after a change of signs on the constraints, whereas the converse implication follows from [26, Th. 47.E, statements (1) and (3)], assuming the Slater condition. □

In case the optimization is carried out over a finite-dimensional space and the constraints are affine, it is well known that the Slater condition is not needed. It turns out that this is the case in infinite dimensions too, as long as the number of constraints is finite; this will be established as item 2) in Theorem 5.

For simplicity, we introduce the notation $\mathbf{P} := \{1, 2, \dots, \widetilde{p}\}$. A vector $\lambda = (\lambda_k)_{k=1}^{\widetilde{p}}$ with the properties in Lemma 4.2) is referred to as a *Lagrange multiplier associated with* z_*. Note that we do not claim that associated Lagrange multipliers are uniquely determined by z_*. The property $\lambda_k f_k(z_*) = 0$ for all k is referred to as *complementarity*. The set of constraints that are active at the optimizer z_*, i.e.,

$$\mathbf{A} := \{k \in \mathbf{P} \mid f_k(z_*) = 0\}$$

is referred to as the *optimal active set*, and its complement is $\mathbf{A}^c := \mathbf{P} \setminus \mathbf{A}$. Denote

$$\mathcal{I}_k := \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{\widetilde{p}}$$

with the one in position $k \in \mathbf{P}$. For any index set $\mathbf{A}$ and operator G mapping into $\mathbb{R}^{\widetilde{p}}$, we use $\#\mathbf{A}$ to denote the number of elements in $\mathbf{A}$, and $G^{\mathbf{A}}$ means G projected onto the components in $\mathbb{R}^{\widetilde{p}}$, which are indexed in $\mathbf{A}$, so that $G^{\mathbf{A}} = \Gamma G$, where $\Gamma = (\mathcal{I}_k)_{k \in \mathbf{A}} \in \mathbb{R}^{(\#\mathbf{A}) \times \widetilde{p}}$, with k in increasing order.

We next apply Lemma 4 to (13), for a fixed parameter value $\theta_n \in X \times U$, in order to get a Hilbert space analog of [1, Th. 2]. The resulting theorem is stronger than most similar results presently in the explicit MPC literature, because we obtain that the minimizer is an affine function of the parameter θ_n, without assuming that the *linear independence constraint qualification (LICQ)* holds at $\mathbf{z}_*(\theta_n)$, i.e., we do not need that $G^{\mathbf{A}}$ is surjective. In fact, if H is invertible, then

$$\begin{aligned} \operatorname{ran}\left(G^{\mathbf{A}} H^{-1} (G^{\mathbf{A}})^*\right) &= \ker\left(G^{\mathbf{A}} H^{-1} (G^{\mathbf{A}})^*\right)^{\perp} \\ &= \ker\left((G^{\mathbf{A}})^*\right)^{\perp} = \operatorname{ran}\left(G^{\mathbf{A}}\right) \end{aligned}$$

so that the LICQ condition holds if and only if the matrix $G^{\mathbf{A}} H^{-1} (G^{\mathbf{A}})^*$ is invertible. In the theorem, we more generally replace this inverse by the pseudoinverse $(G^{\mathbf{A}} H^{-1} (G^{\mathbf{A}})^*)^{[-1]}$, where we denote

$$P^{[-1]} := P\big|_{\ker(P)^{\perp}}^{-1}, \quad \operatorname{dom}\left(P^{[-1]}\right) := \operatorname{ran}(P).$$

Theorem 5: Let $\emptyset \neq \mathbf{A} \subset \mathbf{P}$ be a candidate active set, and let $\lambda := (\lambda_k)_{k=1}^{\widetilde{p}}$ be a column vector of putative Lagrange multipliers. In the notation of (13), the following are true.

1) Assume that $\mathbf{A}$, $\theta_n \in X \times U$ (or $\theta_n \in X$), $\mathbf{z}_* \in U^N$, and $\lambda^{\mathbf{A}} \in \mathbb{R}^{\#\mathbf{A}}$ are such that

$$\begin{aligned} W^{\mathbf{A}} + S^{\mathbf{A}} \theta_n &= G^{\mathbf{A}} \mathbf{z}_* \\ W^{\mathbf{A}^c} + S^{\mathbf{A}^c} \theta_n &\geq G^{\mathbf{A}^c} \mathbf{z}_* \\ \lambda^{\mathbf{A}} \geq 0, \quad H \mathbf{z}_* &= -(G^{\mathbf{A}})^* \lambda^{\mathbf{A}} \end{aligned} \tag{14}$$

where $\mathbf{A}^c := \mathbf{P} \setminus \mathbf{A}$. Then, $\mathbf{z}_*$ is the global minimizer of (13), and the active set at $\mathbf{z}_*$ contains $\mathbf{A}$.

2) Conversely, if $\mathbf{z}_*$ minimizes (13) and $\mathbf{A}$ is the set of all constraints active at $\mathbf{z}_*$, then (14) holds together with $W^{\mathbf{A}^c} + S^{\mathbf{A}^c} \theta_n > G^{\mathbf{A}^c} \mathbf{z}_*$ and $\lambda^{\mathbf{A}^c} = 0$.

3) If H is coercive and (14) holds, then the unique minimizer of (13) is

$$\mathbf{z}_* = H^{-1} (G^{\mathbf{A}})^* \left(G^{\mathbf{A}} H^{-1} (G^{\mathbf{A}})^*\right)^{[-1]} (W^{\mathbf{A}} + S^{\mathbf{A}} \theta_n) \tag{15}$$

and there is some $k \in \ker((G^{\mathbf{A}})^*)$, such that

$$\lambda^{\mathbf{A}} = -\left(G^{\mathbf{A}} H^{-1} (G^{\mathbf{A}})^*\right)^{[-1]} (W^{\mathbf{A}} + S^{\mathbf{A}} \theta_n) \oplus k \tag{16}$$

where $h \oplus k$ denotes $h + k$ with $h \perp k$.

We say that the point $\mathbf{z}_* \in U^N$ is *admissible* if $W + S\theta_n \geq G\mathbf{z}_*$, i.e., $\mathbf{z}_*$ lies in the feasible set $\mathcal{S}(\theta_n)$. We call a candidate active set $\mathbf{A}$, for which there exists some λ that satisfies (14), a *sufficient active set*, since it may be strictly smaller than the set of *all* constraints active at the optimum $\mathbf{z}_*$, but it is nevertheless sufficient for guaranteeing optimality and computing the minimizer.

If LICQ holds at θ_n, then $k = 0$ and the pseudoinverse equals the standard inverse in (15) and (16). Theorem 5 says nothing for $\mathbf{A} = \emptyset$, but it is clear that the optimizer is $\mathbf{z}_* = 0$ in this case, provided that it is admissible, and $\lambda = 0$ works as an associated Lagrange multiplier.

Proof of Theorem 5: Clearly, the assumptions in item 1 (in item 2) imply admissibility of $\mathbf{z}_*$, and hence feasibility of (13). Now, fix a parameter $\theta_n \in X \times U$, such that $\mathcal{S}(\theta_n) \neq \emptyset$.

The functionals $f(\mathbf{z}) := \frac{1}{2} \langle H\mathbf{z}, \mathbf{z} \rangle_{U^N}$ and

$$f_k(\mathbf{z}, \theta_n) := \mathcal{I}_k \left(W + S\theta_n - G\mathbf{z}\right), \qquad 1 \leq k \leq \widetilde{p}$$

are all convex: For all $\mathbf{z}, \mathbf{x} \in U^N$ and $h \in [0, 1]$, using (9) at the end, possibly with $\varepsilon = 0$ unless H is coercive

$$\begin{aligned} &(1-h) f(\mathbf{x}) + h f(\mathbf{z}) - f\left((1-h)\mathbf{x} + h\mathbf{z}\right) \\ &\quad = h \frac{1}{2} \langle H\mathbf{z}, \mathbf{z} \rangle + (1-h) \frac{1}{2} \langle H\mathbf{x}, \mathbf{x} \rangle \\ &\qquad - \frac{1}{2} \langle H(h\mathbf{z} + (1-h)\mathbf{x}), h\mathbf{z} + (1-h)\mathbf{x} \rangle \\ &\quad = h(1-h) \frac{1}{2} \langle H(\mathbf{z} - \mathbf{x}), \mathbf{z} - \mathbf{x} \rangle \\ &\quad \geq \frac{\varepsilon\, h\,(1-h)}{2} \cdot \|\mathbf{z} - \mathbf{x}\|^2 \end{aligned}$$

and $f_k(\cdot, \theta_n)$ is also convex, because

$$h\, f_k(\mathbf{z}, \theta_n) + (1-h)\, f_k(\mathbf{x}, \theta_n) = f_k(h\mathbf{z} + (1-h)\mathbf{x}, \theta_n).$$

These are Fréchet (hence Gâteaux) differentiable on U^N, because

$$\left| f(\mathbf{z} + \mathbf{x}) - f(\mathbf{z}) - \langle H\mathbf{z}, \mathbf{x} \rangle \right| = \frac{|\langle H\mathbf{x}, \mathbf{x} \rangle|}{2} \leq \frac{\|H\|}{2} \cdot \|\mathbf{x}\|^2$$

so that $f'(\mathbf{z}) = H\mathbf{z}$, $\mathbf{z} \in U^N$, and

$$\begin{aligned} &f_k(\mathbf{z} + \mathbf{x}, \theta_n) - f_k(\mathbf{z}, \theta_n) + \langle G^* \mathcal{I}_k^*, \mathbf{x} \rangle = 0 \\ &\text{so that} \quad f_k'(\mathbf{z}, \theta_n) = -G^* \mathcal{I}_k^*, \quad \mathbf{z} \in U^N. \end{aligned} \tag{17}$$

See, for instance, [26, Sec. 40.1].

According to Lemma 4, $\mathbf{z}_*$ solves (13) if $\mathbf{z}_* \in \mathcal{S}(\theta_n)$, and there exist Lagrange multipliers $\lambda_k \geq 0$ with $\lambda_k f_k(\mathbf{z}_*) = 0$ for all $1 \leq k \leq \widetilde{p}$, such that

$$f'(\mathbf{z}_*) = \sum_{k=1}^{\tilde{p}} \lambda_k f_k'(\mathbf{z}_*). \tag{18}$$

By substituting the Fréchet derivatives calculated above, we get that (18) holds if and only if $H\mathbf{z}_* + G^*\lambda = 0$. If $\lambda_k = 0$ for all $k \notin \mathbf{A}$, then $H\mathbf{z}_* + G^*\lambda = 0$ collapses into $H\mathbf{z}_* + (G^{\mathbf{A}})^*\lambda^{\mathbf{A}} = 0$. All constraints in $\mathbf{A}$ are active at the optimum $\mathbf{z}_*$ if and only if $G^{\mathbf{A}}\mathbf{z}_* = W^{\mathbf{A}} + S^{\mathbf{A}}\theta_n$. Hence, (18) holds, $\lambda^{\mathbf{A}^c} = 0$, and all constraints in $\mathbf{A}$ are active if and only if

$$\begin{bmatrix} H & (G^{\mathbf{A}})^* \\ G^{\mathbf{A}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_* \\ \lambda^{\mathbf{A}} \end{bmatrix} = \begin{bmatrix} 0 \\ W^{\mathbf{A}} + S^{\mathbf{A}}\theta_n \end{bmatrix} \tag{19}$$

and $\lambda^{\mathbf{A}^c} = 0$. Assuming additionally that $\lambda^{\mathbf{A}} \in [0,\infty)^{\#\mathbf{A}}$ and $W^{\mathbf{A}^c} + S^{\mathbf{A}^c}\theta_n \geq G^{\mathbf{A}^c}\mathbf{z}_*$, we get from Lemma 4 that $\mathbf{z}_*$ minimizes (13). Hence, whenever (14) holds, we obtain that $\mathbf{z}_*$ minimizes (13) by *defining* $\lambda^{\mathbf{A}^c} := 0$. Noting that some constraints $k \notin \mathbf{A}$ may also be active at $\mathbf{z}_*$, we get that the active set at $\mathbf{z}_*$ is $\mathbf{B}_{\theta_n} \supset \mathbf{A}$. The proof of item 1 is complete.

In order to prove item 2, we will apply [27, Th. 2.4]. For this, we equip all finite sets $\mathcal{F}$ with the discrete topology, so that the σ-algebra of $\mathcal{F}$ equals the power set of $\mathcal{F}$. Thus, all finite sets induce measure spaces where all subsets are measurable, and the standing assumptions in [27, p. 5] are satisfied in a rather trivial manner. By the first paragraph of [27, Sec. 3], [27, Th. 2.4] is applicable, since (13) has only a finite number of constraints, i.e., $\#\mathbf{P} = \widetilde{p} < \infty$.

In order to make use of [27, Th. 2.4], we next need to verify the constraint qualification in [27, Def. 2.2]: For every $\mathbf{z} \in \mathcal{S}(\theta_n)$, and for every nonzero $\mathbf{x} \in U^N$ such that $\langle \mathbf{x}, f_k'(\mathbf{z},\theta_n)\rangle \geq 0$ for all $k \in \mathbf{A}_{\mathbf{z}}$, there exist $\tau > 0$ and a continuous arc $C : [0,\tau) \to U^N$, such that

$$C(0) = \mathbf{z}, \quad C'(0) = \mathbf{x}, \quad \text{and} \quad f_k\left(C(t),\theta_n\right) \geq 0$$

for all $t \in [0,\tau)$ and all $k \in \mathbf{P}$; here, $\mathbf{A}_{\mathbf{z}}$ denotes the set of constraints that are active at $\mathbf{z}$, i.e.,

$$\mathbf{A}_{\mathbf{z}} := \{k \in \mathbf{P} \mid f_k(\mathbf{z},\theta_n) = 0\}.$$

We fix some arbitrary pair $(\mathbf{z},\mathbf{x})$ with the above properties, and we will construct a feasible arc C with the required initial conditions. Since our constraints are affine, we choose $C(t) := \mathbf{z} + t\mathbf{x}$, $t \in [0,\tau)$, as suggested in [27, p. 11]. Then

$$\begin{aligned} f_k\left(C(t),\theta_n\right) &= \mathcal{I}_k\left(W + S\theta_n - G(\mathbf{z}+t\mathbf{x})\right) \\ &= f_k(\mathbf{z},\theta_n) + t\left\langle \mathbf{x}, f_k'(\mathbf{z},\theta_n)\right\rangle \geq 0 \end{aligned} \tag{20}$$

for all $t \geq 0$ and $k \in \mathbf{B}$ with

$$\mathbf{B} := \{k \in \mathbf{P} \mid \langle \mathbf{x}, f_k'(\mathbf{z},\theta_n)\rangle \geq 0\}.$$

Now, let

$$a := \min_{k \in \mathbf{P}\setminus\mathbf{B}} f_k(\mathbf{z},\theta_n) > 0$$

(because $\mathbf{B} \supset \mathbf{A}_{\mathbf{z}}$ by the choice of $\mathbf{z}$ and $\mathbf{x}$) and

$$b := \max_{k \in \mathbf{P}\setminus\mathbf{B}} -\langle \mathbf{x}, f_k'(\mathbf{z},\theta_n)\rangle > 0$$

(because of the definition of $\mathbf{B}$). Then, (20) is satisfied also for all $k \in \mathbf{P}\setminus\mathbf{B}$ and for all $t \in [0,\tau)$, with $\tau := a/b > 0$. Thus, the constraint qualification holds.

Let $\mathbf{z}_*$ be a minimizer of (13) with the corresponding active set $\mathbf{A}$. By [27, Th. 2.4], the equivalent conditions in [27, Prop. 2.2] hold, i.e., there exists a finite measure u^* on $\mathbf{P}$, such that we have the following properties.

1) In a sense made precise in [27], we have

$$f'(\mathbf{z}_*) = \int_{\mathbf{P}} f_k'(\mathbf{z}_*,\theta_n)\, du^*. \tag{21}$$

2) $u^*(\mathbf{P}') = 0$ for all $\mathbf{P}' \subset \mathbf{A}^c$.
3) $u^*(\mathbf{P}') \geq 0$ for all $\mathbf{P}' \subset \mathbf{P}$.

Next, define $\lambda_k := u^*(\{k\})$ for $k \in \mathbf{P}$. By properties 2 and 3 of u^*, $\lambda_k = 0$ for all $k \in \mathbf{A}^c$ and $\lambda_k \geq 0$ for all $k \in \mathbf{A}$. Since $\mathbf{P}$ is finite, the integral (21) simplifies, using (17) and the corresponding expression for f', to

$$H\mathbf{z}_* = \sum_{k\in\mathbf{P}} -G^*\mathcal{I}_k^* u^*(\{k\}) = -G^*\lambda = -(G^{\mathbf{A}})^*\lambda^{\mathbf{A}}$$

because $\lambda^{\mathbf{A}^c} = 0$. This completes the proof of item 2.

Finally, for item 3, note that by (14), the problem is feasible, and then, Lemma 3 gives (the existence and) the uniqueness of the minimizer $\mathbf{z}_*$. The conditions (14) in particular imply (19), and solving the latter for $\lambda^{\mathbf{A}}$, we get (16), and then, (15) follows easily. □

Due to Theorem 5, the dual-active-set method QPKWIK algorithm, the basis of the `mpcActiveSetSolver` algorithm in MATLAB, is applicable to solving the strictly convex quadratic programming problem (13). This method is too involved to be reproduced here, but some of its nice properties are as follows.

1) It is easy to find a dually feasible initial point for the iteration, namely, the unconstrained optimum.
2) LICQ is in general not maintained throughout optimization, but one has more leeway in avoiding LICQ-related problems than is the case with primal active set methods.
3) It solves (13) or determine that no solution exists in a finite number of steps, since $\widetilde{p} < \infty$, and since each candidate active set is visited at most once.

We close this section by observing that the mentioned algorithm in general operates only with *sufficient* active sets rather than *full optimal active set*.

V. Example: Explicit MPC of a Timoshenko Beam

A. Port-Hamiltonian Formulation of a Timoshenko Beam

As in [28, Ex. 2.19] (see also [29, Ex. 7.1.4]), we consider the Timoshenko beam model on the spatial interval $\xi \in [0,1]$. Let $w(\xi,t)$ and $\phi(\xi,t)$ denote the transverse displacement and the rotation angle of the beam, respectively, and let $\rho, I_\rho, EI, K \in C^1(0,1;\mathbb{R}_+)$ denote mass per unit length, the rotary moment of inertia of the cross section, the product of Young's modulus of elasticity and the moment of inertia of the cross section, and the shear modulus, respectively. By defining a state variable $x = (x_1,x_2,x_3,x_4) := (w' - \phi, \rho\dot{w}, \phi', I_\rho\dot{\phi})$, where $()'$ and $\dot{()}$ denote spatial and temporal derivatives, respectively, the beam model can be written as a port-Hamiltonian system

$$\dot{x}(t) = P_1\left(\mathcal{H}x(t)\right)' + P_0\mathcal{H}x(t), \quad x(0) = x_0 \tag{22}$$

where $\mathcal{H} = \operatorname{diag}(K, 1/\rho, EI, 1/I_\rho)$ and

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We consider a cantilever beam, where we control the free end. That is, assuming that the end at $\xi = 0$ is clamped, the boundary conditions and controls of (22) are given by

$$\frac{1}{\rho(0)}x_2(0,t) = \frac{1}{I_\rho(0)}x_4(0,t) \equiv 0 \tag{23a}$$

$$u_1(t) = K(1)x_1(1,t) \tag{23b}$$

$$u_2(t) = EI(1)x_3(1,t). \tag{23c}$$

By checking the conditions of [29, Th. 13.2.2], we have that the system (22) and (23) is well-posed on $X = L^2(0,1;\mathbb{R}^4)$, i.e., it has a well-defined unique weak solution for any inputs $u_1, u_2 \in L^2_{loc}(0,\infty;\mathbb{R})$ and initial conditions $x_0 \in X$.

B. Cost Functional and Conversion to Discrete Time

Consider the cost functional

$$\int_0^\infty \langle Q'x(t), x(t)\rangle + \langle R'u(t), u(t)\rangle + \langle V'\dot{u}(t), \dot{u}(t)\rangle \, dt \tag{24}$$

with weights $Q', R', V' \geq 0$ and with constraints $\int_0^1 x_1(t)dt \leq \overline{x}_1$, $\int_0^1 x_4(t)dt \geq \underline{x}_4$, and $\underline{u} \leq u_{1,2}(t) \leq \overline{u}$ for all $t \geq 0$, where the bounds $\overline{x}_1, \underline{x}_4, \underline{u}$, and $\overline{u}$ will be specified in Section V-C. Next, we will convert the continuous-time problem (24) into discrete-time problem and formulate the MPC procedure to find a control signal that approximately minimizes (24).

The system (22) and (23) is transformed to discrete time using the Cayley transform [30, eq. (1.5)]. For the boundary control system (22) and (23), the Cayley transform can be computed explicitly based on the Laplace transform of (22) and [31, Rem. 10.1.5] similar to [24, Sec. 4.1]. However, the closed-form expressions of the discrete-time operators would be lengthy, so for practical computations and computational efficacy, we prefer computing the discrete-time operators based on finite-dimensional approximations of (22) and (23). We will specify the employed approximation methods in Section V-C.

Converting the cost function (24) into discrete time, the coefficients Q', R', and V' should be scaled by the time discretization interval h to account for the temporal integration (and to make the cost function independent of h). However, we need to take into account that $u_k/\sqrt{h}$ approximates $u(t)$ in the Cayley transform. Thus, the discrete-time cost function on some finite horizon $[0, N-1]$ is given by

$$\sum_{k=0}^{N-1} \big(\langle Qx_k, x_k\rangle + \langle Ru_k, u_k\rangle + \langle V(u_k - u_{k-1}), u_k - u_{k-1}\rangle\big) \tag{25}$$

where $Q = hQ'$, $R = R'$, and $V = h^{-2}V'$ (scaling by h^{-2} comes from the finite-difference approximation of $\dot{u}(t)$). Compared to (3), we have taken $V_N = 0$ as we see no reason to additionally penalize $u_{N-1} \neq 0$, albeit for longer prediction horizons, this has very little impact on the solution. Moreover, even if we have not added any terminal ingredients to (25), based on the simulation results of the next section, the MPC procedure is stabilizing. While stability of MPC is outside of our scope, we note that the turnpike property of *unconstrained* quadratic problems of the form (24) (with $V = 0$) has been considered in [32], which would provide one way of guaranteeing stability of MPC without terminal ingredients. However, these results cannot be (directly) applied to *constrained* problems (with $V > 0$), so it is unclear whether the constrained minimization problem of (24) has the turnpike property or not.

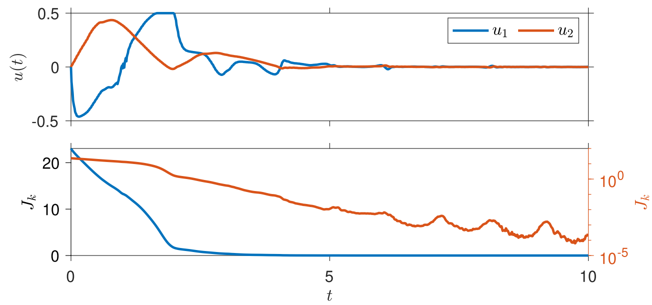

Fig. 1. Controls and the optimal costs (logarithmic scale on the right).

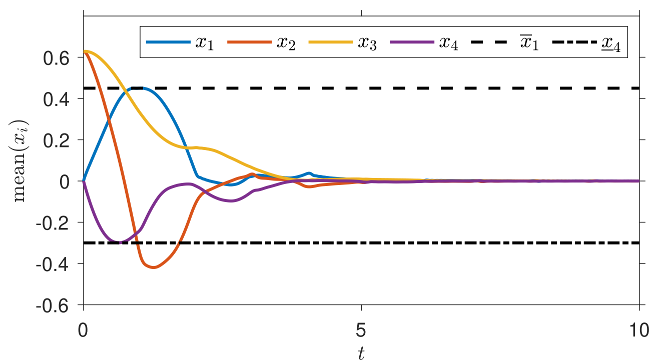

Fig. 2. Means of the plant state components and the state constraints.

C. Numerical Simulation

In the simulation, the input constraints are set to $u_{1,2} \in [-0.5, 0.5]$ and the state constraints are $\underline{x}_4 = -0.3$ and $\overline{x}_1 = 0.45$. The weights in the cost function are $Q' = 100$, $R = 1$, and $V' = 0.1$, and the time discretization is $h = 2^{-7}$. The MPC prediction horizon is set to $N = 30$. For simplicity, the physical parameters of the beam are set to 1. The initial conditions are given by $x_1 = x_4 = 0, x_2 = \sin(\frac{\pi}{2}\xi)$, and $x_3 = \cos(\frac{\pi}{2}\xi)$, and we initialize $u_{-1} = 0$.

In order to demonstrate a somewhat realistic scenario, where the prediction model is not a perfect copy of the system dynamics, we employ different approximations for the plant and the prediction model. For the plant, we approximate the beam model by finite differences, and for the prediction model, we employ a spectral Galerkin approximation. In the finite-difference approximation of the plant, we use 127 grid points, and in the spectral Galerkin approximation, we use nine polynomial basis functions. Thus, the prediction model is lower dimensional by a factor of 14, which emulates the difference between an infinite-dimensional plant and a finite-dimensional prediction model. Moreover, the continuous-time nature of the plant is emulated by simulating the plant using the `ode45` solver in MATLAB as opposed to using a prescribed time discretization in the simulation.

The simulation results are displayed in Figs. 1–3. First, Fig. 1 displays the optimal controls solved using the `mpcActiveSetSolver` in MATLAB and the corresponding optimal costs. It can be seen that the controls satisfy the input constraints and that the optimal costs are (exponentially) decreasing during the first part of the simulation. However, the evolution of the optimal costs experiences some ripples toward the end, even if the long-term trend is still decreasing, likely due to the imperfections in the prediction model. Moreover, the cost function values are already very close to zero when the ripples occur.

Fig. 2 shows the mean values of the state components over the spatial interval along with the state constraints. While it appears that the state

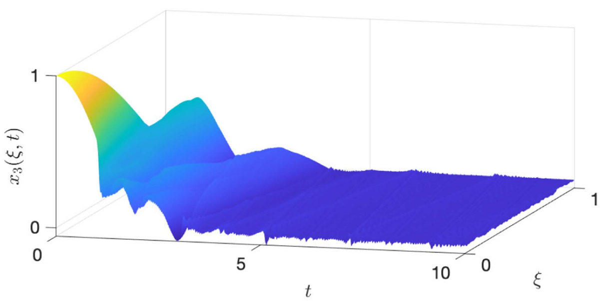

Fig. 3. Profile of the plant state component $x_3 = \phi'$.

TABLE I
COMPARISON OF COMPUTATIONAL TIMES AND CONTROL COSTS FOR DIFFERENT PREDICTION HORIZON LENGTHS N

N	time (s)	J_d	$2^{\tilde{p}}$
10	0.34	148	2.88×10^{17}
20	0.54	126	3.32×10^{35}
30	0.84	122	2.83×10^{53}
40	1.34	121	4.41×10^{71}
50	2.20	120	5.06×10^{89}
60	3.38	119	5.87×10^{107}
70	5.34	119	6.77×10^{125}

constraints become active at some point, a closer inspection of Fig. 2 reveals that the constraint $\overline{x}_1$ in fact gets violated (very slightly) as the maximum value of mean(x_1) is 3.78×10^{-4} larger than the upper bound $\overline{x}_1 = 0.45$. This violation of the state constraint is very mild, and we view the constraint as still being satisfied in the soft sense [1, Sec. 6.3]. We note that strict satisfaction of state constraints under imperfect prediction model would require robustness considerations, which are outside the scope of this study. Finally, we note that $\underline{x}_4$ does not strictly get activated as the minimum value of mean(x_4) is 7.61×10^{-4} larger than the lower bound $\underline{x}_4 = -0.3$.

For additional illustration, we also present the state profile of the plant component x_3 in Fig. 3, where the effect of an imperfect prediction model can also be seen. Somewhat similar to the cost function values in Fig. 1, small ripples appear in the state profile as the value approaches zero, albeit these may also be caused by numerical noise. Interestingly enough, these ripples did not appear in Fig. 2—possibly due to the averaging effect of the mean value. Regardless, the state component seems to remain in some δ-neighborhood of zero once it enters there, which can viewed as a version of practical stability [15, Def. 2.15].

To conclude this section, we briefly comment on the choice of the prediction horizon N. In Table I, we compare the computational times and optimal costs of the `mpcActiveSetSolver` for different horizon lengths. It can be seen that the algorithm is very efficient, even if the computational time increases along with the horizon length, arguably due to the exponentially growing number of candidate sets shown in the rightmost column of the table. Regardless, even the longest tested horizon 70 is easily feasible for the ten-second simulation run. One explanatory factor in the rapid performance of the algorithm may be that we use the optimal active set of the previous control step as the initial guess for the next one, and if this set does not change much between steps—which often seems to be the case in this simulation—the algorithm finds an optimal active set already after a few iterations.

Table I additionally shows that the optimal cost decreases with the optimization horizon length, as expected. However, the optimal control costs J_d do not noticeably change after the shortest horizon $N = 10$, nor would Figs. 1–3. Moreover, with longer prediction horizons, the imperfections may accumulate, and hence, the late-horizon predictions may become unreliable. Hence, one might want to prioritize the early-horizon predictions with time-varying weights in the cost functional or simply employ a shorter prediction horizon as we did in the simulation.

ACKNOWLEDGMENT

The authors would like to gratefully acknowledge the input of the audience at the 2022 International Symposium on Mathematical Theory of Networks and Systems, and Prof. Martin Mönnigmann, Prof. Mark Cannon, Prof. Eric Kerrigan, and Prof. Tor Arne Johansen, for enlightening discussions during the preparation of this manuscript. An anonymous referee pointed out [27], leading to a major improvement of Theorem 5.

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