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On the maximum increase and decrease of one-dimensional diffusions

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Abstract

In this paper the joint distribution of the maximum increase and the maximum decrease up to a first hitting time is calculated for a regular one-dimensional diffusion. Moreover, it is shown that the process given by the maximum decrease when the hitting level is the "time" parameter is a pure jump Markov process and its generator is found. As examples, Brownian motion and three dimensional Bessel process are analyzed more in detail.

Keywords: running maximum, running minimum, maximum drawdown, maximum drawup, optional sampling, $h-$transform, Brownian motion with drift, geometric Brownian motion.

AMS Classification: 60J60, 60J65, 60G17, 62P05.

1 Introduction

We start with some notation. Let $X = (X_t)_{t\geq 0}$ be a regular one-dimensional diffusion taking values on an arbitrary open interval $I = (l, r) \subseteq \mathbf{R}$ in the sense of Itô and McKean [6], see [2]. It is assumed that the process does not die inside (l, r) and is killed if it hits l or r. Recall that X is a strong Markov process with continuous sample paths. We let $\mathbf{P}_x$ and $\mathbf{E}_x$ denote the probability measure and the expectation operator associated with X when $X_0 = x \in I$. The standard notation $\mathcal{F}_t$ is used for the σ-algebra generated by X up to time $t \geq 0$, and we set $\mathcal{F} := \mathcal{F}_\infty$. We let m and S stand for the speed measure and the scale function of X, respectively. For the boundary classification in terms of m and S, see [2].

The **maximum increase** up to time t is defined by

$$D_t^+ := \sup_{0\leq u\leq v\leq t} (X_v - X_u)\,, \tag{1.1}$$

and the **maximum decrease** up to time t by

$$D_t^- := \sup_{0\leq u\leq v\leq t} (X_u - X_v)\,. \tag{1.2}$$

Notice that defining the running maximum and minimum via

$$M_t := \sup_{s\leq t} X_s, \qquad I_t := \inf_{s\leq t} X_s, \tag{1.3}$$

respectively, we may write

$$D_t^+ = \sup_{s\leq t}(X_s - I_s), \qquad D_t^- = \sup_{s\leq t}(M_s - X_s).$$

One motivation – in addition to pure mathematical curiosity – to study the maximum decrease and increase comes from mathematical finance. In this application, the diffusion X is the model for the price process of the underlying asset. The maximum decrease, also called maximum drawdown (MDD), is used in portfolio management to quantify the riskyness of a stock or some other asset. Related measures to MDD often used are e.g. the recovery time from MDD and the duration of MDD. Similar quantities can be defined for the maximum increase, that is, for the maximum drawup, MDU. For more details about applications in finance, we refer to [11], [9],

[1], [13], [14], [20], and [4]. For Lévy processes and the maximum drawdown, see also [12]

For Brownian motion the joint distribution of D_T^- and D_T^+, where T is an independent exponential time, is computed in [19], see Proposition 4.6. We refer to [19] also for the background of the problem setting and further references.

In an intermediate step of the calculations in [19] we apply the Brownian excursion theory to find the law of the maximum decrease up to a hitting time. Hereby the spatial homogeneity is of key importance, and the approach cannot be used for other regular diffusions than Brownian motion (with drift). In the present paper it is seen that the martingale theory provides tools such that the distribution can be found for a general diffusion. Notice, however, that the excursion approach applies for spectrally negative Lévy processes, see [21].

Notice also the following connection with the notion of range:

$$R_t := M_t - I_t = \sup_{0\le u,v\le t}(X_v - X_u) = \max\{D_t^+, D_t^-\}.$$

The range for stochastic processes has been much studied starting from the works of P. Lévy and W. Feller. For further references and results , see [5], [22] and [18].

This paper is organised as follows. In Section 2 we focus on the distribution of $D_{H_\beta}^-$, where H_β is the first hitting time of level $\beta \in I$. In Section 3 the joint distribution of $D_{H_\beta}^-$ and $D_{H_\beta}^+$ is determined. In particular, it is seen that the distribution depends only on the scale function of the diffusion. In Section 4, we prove that the maximum decrease process $(D_{H_\beta}^-)_{\beta\ge 0}$ is a pure jump Markov process, find its generator and transition probabilities.

2 Distribution of the maximum decrease

Let $X = (X_t)_{t\ge 0}$ be a regular diffusion on an open interval (l, r) as introduced in Section 1 and define the first hitting time of $\beta \in (l, r)$ via

$$H_\beta := \begin{cases} \inf\{t \,:\, X_t = \beta\}, & \text{if } \{\cdot\} \neq \emptyset, \\ +\infty & \text{if } \{\cdot\} = \emptyset. \end{cases} \tag{2.1}$$

We also define $H_r := \lim_{\beta\uparrow r} H_\beta$ and $H_l := \lim_{\beta\downarrow l} H_\beta$.

The main task in this section is to calculate the $\mathbf{P}_x$-distribution of the maximum decrease up to H_β given by

$$D^-_{H_\beta} := \sup_{0\le s\le H_\beta} (M_s - X_s).$$

Here we separate two cases depending on whether $x > \beta$, Proposition 2.2, or $x < \beta$, Proposition 2.3. The distribution for the maximum increase can be derived using analogous methods as for the maximum decrease. Notice also that the maximum increase for X equals to the maximum decrease for $-X$.

To simplify the presentation we make the following assumption

(A) $\quad S(l) = -\infty$

taken to be valid – if nothing else is said – for the rest of the paper. Recall that under (A) it holds

(A1) If $S(r) < +\infty$ then either $H_r < \infty$ a.s. or $\lim_{t\to\infty} X_t = r$ a.s.,

(A2) If $S(r) = +\infty$ then $\limsup_{t\to\infty} X_t = r$ and $\liminf_{t\to\infty} X_t = l$ a.s.,

(A3) $H_l = +\infty$ a.s.,

(A4) $H_\beta < +\infty$ $\mathbf{P}_x$-a.s. for all $x \le \beta$.

Remark 2.1. *To transform the results derived below to the corresponding case with $S(r) = +\infty$ one may consider the process $-X$. The details are left to the reader.*

The $\mathbf{P}_x$-distribution of the maximum decrease up to H_β in case $x > \beta$ is essentially given by the distribution of the maximum before time H_β. This is made precise in the proof of our first result. Notice that here assumption (A) is not needed.

Proposition 2.2. *For $x > \beta$ and $u > 0$ it holds*

$$\mathbf{P}_x(D^-_{H_\beta} > u\,;\, H_\beta < +\infty) = \tag{2.2}$$

$$= \begin{cases} 0, & u \ge r - \beta \\ \frac{S(x)-S(\beta)}{S(u+\beta)-S(\beta)}\,\frac{S(r)-S(u+\beta)}{S(r)-S(\beta)}, & x-\beta \le u < r-\beta \\ \frac{S(r)-S(x)}{S(r)-S(\beta)}, & u < x-\beta, \end{cases}$$

where the convention $\infty/\infty = 1$ is used in case $S(r) = +\infty$, and if $r = +\infty$ the first case above is absent.

Proof. Firstly, for $u \geq r - \beta$ the asked probability is 0 because

$$D^-_{H_\beta} = M_{H_\beta} - X_{H_\beta} = M_{H_\beta} - \beta \leq r - \beta$$

on $\{H_\beta < +\infty\}$. Secondly, if $x - \beta \leq u < r - \beta$ we have by the strong Markov property

$$\begin{aligned}\mathbf{P}_x(D^-_{H_\beta} > u\,;\, H_\beta < +\infty) &= \mathbf{P}_x(D^-_{H_\beta} > u\,;\, H_\beta < H_r)\\ &= \mathbf{P}_x(H_{u+\beta} < H_\beta)\mathbf{P}_{u+\beta}(H_\beta < H_r),\end{aligned}$$

and, thirdly, for $u < x - \beta$

$$\mathbf{P}_x(D^-_{H_\beta} > u\,;\, H_\beta < H_r) = \mathbf{P}_x(H_\beta < H_r).$$

Using the well known properties of the scale function, see [2] p. 14, the claimed formula results. □

Next we consider the case $x < \beta$. The proof in this case is more challenging since it is not possible to connect $D^-_{H_\beta}$ with the minimum of X up to H_β. We state next the result but the proof is given after some lemmas.

Proposition 2.3. *The $\mathbf{P}_x$-distribution of the maximum decrease up to H_β is given for $x < \beta$ by*

$$\mathbf{P}_x(D^-_{H_\beta} < u) = \begin{cases} 1, & u \geq \beta - l, \\ \exp\left(-\int_{u+l}^{\beta} \frac{S(dy)}{S(y)-S(y-u)}\right), & x - l \leq u \leq \beta - l. \\ \exp\left(-\int_{x}^{\beta} \frac{S(dy)}{S(y)-S(y-u)}\right), & 0 \leq u \leq x - l. \end{cases} \tag{2.3}$$

In case $l = -\infty$ the two first cases are absent.

Introduce a new process $Y = (Y_t)_{t\geq 0}$ by setting $Y_t := S(X_t)$. Then, since S is continuous and increasing, Y is a regular diffusion in natural scale and a local martingale taking values in $(S(l), S(r))$. Moreover, define

$$N_t := \sup_{0\leq s\leq t} Y_s = S(M_t) \tag{2.4}$$

and $N_\infty := \lim_{t\to+\infty} N_t$. Let f be a measurable, bounded, and non-negative function with compact support in $[S(x), S(r))$, and set

$$F(y) := \int_{S(x)}^{y} f(t)dt, \qquad y \geq S(x).$$

Since Y is a continuous local martingale the integration by parts (for a formulation well adapted to this case, see Proposition 3.1 in [16]) yields the following result.

Lemma 2.4. *The process $Z = (Z_{t\wedge H_r})_{t\geq 0}$ given for $t < H_r$ by*

$$Z_t := f(N_t)(N_t - Y_t) - F(N_t)$$

is a local $\mathcal{F}_t$-martingale, where, e.g., $Y_{H_r} := S(r)$ if $H_r < +\infty$.

For the discussion to follow, introduce

$$R_u^- := \inf\{t\,:\, M_t - X_t = u\} = \inf\{t\,:\, D_t^- = u\}$$

with the usual convention $\inf\emptyset = +\infty$. Recall also that

$$\{R_u^- < +\infty\} = \{R_u^- < H_r\}.$$

Lemma 2.5. *For f as above and for $0 < u < x - l$ it holds*

$$\mathbf{E}_x\left(f(N_{R_u^-})\left(N_{R_u^-} - S(S^{-1}(N_{R_u^-}) - u)\right)\mathbf{1}_{\{R_u^- < H_r\}}\right) \\ = \mathbf{E}_x(F(N_{R_u^-\wedge H_r})). \tag{2.5}$$

Proof. We show first that $(Z^o_{t\wedge R_u^-})_{t\geq 0}$ is bounded a.s., where, to simplify the notation, $Z_t^o := Z_{t\wedge H_r}$. Recall that $M_t \to r$ a.s. as $t \to +\infty$. From the assumptions on f it follows for all $t > 0$

$$0 \leq f(N_t) \leq C_2$$

and

$$0 \leq F(N_t) \leq C_1 \tag{2.6}$$

for some constants C_1 and C_2, where we used that $N_t = S(M_t) \in [S(x), S(r))$. Clearly, if $t \leq R_u^-$ then $M_t - X_t \leq u$ and

$$X_t \geq M_t - u \geq x - u > l,$$

where the second inequality holds since $X_0 = x$. Hence,

$$Y_t = S(X_t) \geq S(x-u) \tag{2.7}$$

and since $N_t - Y_t \geq 0$ we obtain, using again that f has a compact support, for $t \leq R_u^-$

$$0 \leq f(N_t)(N_t - Y_t) \leq f(N_t)(N_t - S(x-u)) \leq C_3 \tag{2.8}$$

with some constant C_3. Consequently, by (2.6) and (2.8) the boundedness claim is proved. Since $(Z^o_{t\wedge R_u^-})_{t\geq 0}$ is a (bounded) martingale we have for all $t \geq 0$

$$\mathbf{E}_x(Z^o_{t\wedge R_u^-}) = \mathbf{E}_x(Z^o_0) = 0. \tag{2.9}$$

But (2.6) and (2.8) imply that (2.9) is equivalent with

$$\mathbf{E}_x\left(f(N_{t\wedge R_u^-\wedge H_r})\left(N_{t\wedge R_u^-\wedge H_r} - Y_{t\wedge R_u^-\wedge H_r}\right)\right) = \mathbf{E}_x(F(N_{t\wedge R_u^-\wedge H_r})).$$

Letting $t \to +\infty$ and using the dominated convergence theorem which is justified by (2.8) and (2.6) gives

$$\mathbf{E}_x\left(f(N_{R_u^-})\left(N_{R_u^-} - Y_{R_u^-}\right)\mathbf{1}_{\{R_u^- < H_r\}}\right) = \mathbf{E}_x(F(N_{R_u^-\wedge H_r})), \tag{2.10}$$

where it is also used that

$$f(N_{H_r})(N_{H_r} - Y_{H_r}) = 0,$$

which holds since $N_{H_r} = S(r)$ and f has a compact support included in $[S(x), S(r))$. It remains to show that if $R_u^- < H_r$

$$Y_{R_u^-} = S\big(S^{-1}(N_{R_u^-}) - u\big).$$

The continuity of $t \mapsto M_t - X_t$ yields

$$M_{R_u^-} - X_{R_u^-} = u. \tag{2.11}$$

Recall

$$M_{R_u^-} = S^{-1}(N_{R_u^-}),$$

and combining this with (2.11) yields

$$X_{R_u^-} = S^{-1}(N_{R_u^-}) - u,$$

and, further,

$$Y_{R_u^-} = S(X_{R_u^-}) = S(S^{-1}(N_{R_u^-}) - u),$$

and putting this in (2.10) proves the claim. □

Remark 2.6. *Notice that (2.5) does not necessarily hold in case $l > -\infty$ if $u = x - l$ since the right hand side in (2.7) equals $-\infty$.*

Proof of Proposition 2.3 Let ν_x denote the probability distribution associated with $N_{R_u^-}$ under $\mathbf{P}_x$. We assume first that $u < x - l$. Under this condition the relation (2.5) is valid and takes in terms of ν_x the form

$$\int_{[S(x),S(r))} f(y) \left(y - S(S^{-1}(y) - u)\right) \nu_x(dy) = \int_{[S(x),S(r)]} F(y)\, \nu_x(dy), \quad (2.12)$$

where it is also used that a.s.

$$\{R_u^- < H_r\} = \{M_{R_u^-} < r\} = \{S(M_{R_u^-}) < S(r)\} = \{N_{R_u^-} < S(r)\}.$$

Using the definition of F and Fubini's theorem the right hand side of (2.12) is rewritten as

$$\int_{[S(x),S(r)]} F(y)\, \nu_x(dy) = \int_{S(x)}^{S(r)} f(y)\, \nu_x([y, S(r)])\, dy. \quad (2.13)$$

Since f is a non-negative function with a compact support included in $[S(x), S(r))$ it follows from (2.12) and (2.13)

$$\left(y - S(S^{-1}(y) - u)\right) \nu_x(dy) = \nu_x([y, S(r)])\, dy, \quad y \in [S(x), S(r)),$$

and, consequently,

$$\frac{\nu_x(dy)}{\nu_x([y, S(r)])} = \frac{dy}{y - S(S^{-1}(y) - u)}. \quad (2.14)$$

Solving (2.14) gives for $y \geq S(x)$

$$\begin{aligned} \mathbf{P}_x(N_{R_u^-} \geq y) &= \nu_x([y, S(r)]) \\ &= \exp\left(-\int_{S(x)}^{y} \frac{dz}{z - S(S^{-1}(z) - u)}\right) \\ &= \exp\left(-\int_{x}^{S^{-1}(y)} \frac{S(dz)}{S(z) - S(z-u)}\right). \end{aligned} \quad (2.15)$$

Finally, to complete the proof in case $u < x - l$ and $x \leq \beta < r$ observe that

$$\{D_{H_\beta}^- < u\} = \{H_\beta < R_u^-\} = \{\beta < M_{R_u^-}\} = \{S(\beta) < N_{R_u^-}\}, \quad (2.16)$$

where $R_u^- = +\infty$ is allowed with $M_\infty := r$ and $N_\infty := S(r)$. Consequently, from (2.15)

$$\mathbf{P}_x(D_{H_\beta}^- < u) = \mathbf{P}_x(S(\beta) < N_{R_u^-}) = \exp\left(-\int_x^\beta \frac{S(dz)}{S(z) - S(z-u)}\right). \quad (2.17)$$

By monotone convergence, (2.17) is valid also for $u = x - l$ when $l > -\infty$. Next we assume that $l > -\infty$ and $x - l < u < \beta - l$. Notice that (2.16) holds true also in this case. Consequently, to find the $\mathbf{P}_x$-distribution of $D_{H_\beta}^-$ is equivalent to finding the $\mathbf{P}_x$-distribution of $M_{R_u^-}$. For the latter calculation observe first that on $\{R_u^- < +\infty\}$ we have, since $X_{R_u^-} > l$,

$$M_{R_u^-} = X_{R_u^-} + u > l + u.$$

Hence, on $\{R_u^- \leq +\infty\}$ with $X_0 = x < l + u$ and recalling that $H_{l+u} < +\infty$ $\mathbf{P}_x$-a.s. it holds

$$R_u^- > H_{l+u} \quad \text{a.s.}$$

Therefore, for all x, u, and β such that $x - l < u < \beta - l$

$$\begin{aligned}\mathbf{P}_x(M_{R_u^-} > \beta) &= \mathbf{P}_x(M_{R_u^-} > \beta\,;\, H_{l+u} < R_u^-) \\ &= \mathbf{P}_x(M_{R_u^-} \circ \theta_{H_{l+u}} > \beta\,;\, H_{l+u} < R_u^-) \\ &= \mathbf{P}_{l+u}(M_{R_u^-} > \beta), \end{aligned} \quad (2.18)$$

where $(\theta_t)_{t\geq 0}$ is the shift operator and the strong Markov property is applied. Consequently, evoking (2.16) we may use (2.17) with $x = u + l$ yielding

$$\mathbf{P}_x(D_{H_\beta}^- < u) = \exp\left(-\int_{u+l}^\beta \frac{S(dz)}{S(z) - S(z-u)}\right), \quad x - l \leq u \leq \beta - l,$$

and this completes the proof. □

3 Joint distribution

In the next proposition we give the joint distribution of $D_{H_\beta}^-$ and $D_{H_\beta}^+$ when the underlying diffusion starts below β. At the end of the section some examples are discussed. To formulate the result when the starting state is larger than β is left to the reader.

Proposition 3.1. *For $x < \beta \le r$ it holds*
(i) *if* $0 < \beta - x \le u < \beta - l$ *and* $0 \le v < \beta - l$

$$\mathbf{P}_x(D^-_{H_\beta} < v\,;\, D^+_{H_\beta} < u) = \mathbf{P}_x(D^-_{H_\beta} < v\,;\, H_\beta < H_{\beta-u}) \tag{3.1}$$
$$= \begin{cases} \frac{S(x)-S(\beta-u)}{S(\beta)-S(\beta-u)}, & u \le v, \\ \\ \frac{S(x)-S(\beta-u)}{S(v+\beta-u)-S(\beta-u)} \exp\left(-\int_{v+\beta-u}^{\beta} \frac{S(dy)}{S(y)-S(y-v)}\right), & x-\beta+u \le v \le u, \\ \\ \exp\left(-\int_x^\beta \frac{S(dy)}{S(y)-S(y-v)}\right), & 0 \le v \le x-\beta+u, \end{cases}$$

(ii) *if* $u \le \beta - x$

$$\mathbf{P}_x(D^-_{H_\beta} < v\,;\, D^+_{H_\beta} < u) = 0, \tag{3.2}$$

(iii) *if* $u > \beta - l$

$$\mathbf{P}_x(D^-_{H_\beta} < v\,;\, D^+_{H_\beta} < u) = \mathbf{P}_x(D^-_{H_\beta} < v), \tag{3.3}$$

(iv) *if* $v > \beta - l$

$$\mathbf{P}_x(D^-_{H_\beta} < v\,;\, D^+_{H_\beta} < u) = \mathbf{P}_x(D^+_{H_\beta} < u). \tag{3.4}$$

Proof. Statements (ii), (iii) and (iv) are fairly obvious by the definitions of $D^+_{H_\beta}$ and $D^-_{H_\beta}$. Notice that in case $l = -\infty$ statements (iii) and (iv) are absent. The right hand side of (3.4) can be calculated as the corresponding formula (2.2) for $D^-_{H_\beta}$ in Proposition 2.2. The result is

$$\mathbf{P}_x(D^+_{H_\beta} > u) = \begin{cases} 0, & u \ge \beta - l \\ \frac{S(\beta)-S(x)}{S(\beta)-S(\beta-u)}, & \beta - x \le u < \beta - l \\ 1, & u < \beta - x. \end{cases}$$

Consider now statement (i). The first equality in (3.1) is clear since $D^+_{H_\beta} = \beta - I_{H_\beta}$. For the second one we write

$$\mathbf{P}_x(D^-_{H_\beta} < v\,;\, H_\beta < H_{\beta-u}) \tag{3.5}$$
$$= \mathbf{P}_x(D^-_{H_\beta} < v \,|\, H_\beta < H_{\beta-u})\mathbf{P}_x(H_\beta < H_{\beta-u}).$$

The basic property of the scale function says that

$$\mathbf{P}_x(H_\beta < H_{\beta-u}) = \frac{S(x)-S(\beta-u)}{S(\beta)-S(\beta-u)}. \tag{3.6}$$

Since $\beta - u < x < \beta$ the conditional probability in (3.5) can be calculated by studying the process X killed when it exits the interval $(\beta - u, \beta)$ and conditioned to exit at β. We let $\widehat{X}$ denote this conditioned process. Then $\widehat{X}$ can be realized as Doob's h-transform of X (see [19] Lemma 2.3 for such a construction when X is a Brownian motion with drift). The excessive function h to be used is

$$h(x) := \mathbf{P}_x(H_\beta < H_{\beta-u}).$$

In particular, $\widehat{X}$ is a diffusion on the open interval $(\beta - u, \beta)$ with the scale function $S^{(h)}$ given by

$$S^{(h)}(dz) = (S(z) - S(\beta - u))^{-2}\, S(dz), \quad z \in (\beta - u, \beta).$$

We refer to [2] II.31 p. 34 for general formulas for the scale function and the speed measure of an h-transform. These formulas go back to [10], [23], and [7], see also [17], and for detailed proofs, in the general case, [8] and [3]. Straightforward calculations show that we may take

$$S^{(h)}(z) = -(S(z) - S(\beta - u))^{-1}, \quad z \in (\beta - u, \beta). \tag{3.7}$$

Clearly,

$$S^{(h)}(\beta - u) = -\infty \quad \text{and} \quad S^{(h)}(\beta) < +\infty.$$

Let $\mathbf{P}_x^{(h)}$, $x \in (\beta - u, \beta)$, denote the probability measure associated with $\widehat{X}$. Then it holds for $v \geq 0$

$$\mathbf{P}_x(D^-_{H_\beta} < v \,|\, H_\beta < H_{\beta-u}) = \mathbf{P}_x^{(h)}(D^-_{H_\beta} < v), \tag{3.8}$$

and applying (2.3) for $\widehat{X}$ yields

$$\mathbf{P}_x^{(h)}(D^-_{H_\beta} < v) = \begin{cases} 1, & v \geq u, \\ \exp\left(-\int_{v+\beta-u}^{\beta} \frac{S^{(h)}(dy)}{S^{(h)}(y) - S^{(h)}(y-v)}\right), & x - \beta + u \leq v \leq u, \\ \exp\left(-\int_{x}^{\beta} \frac{S^{(h)}(dy)}{S^{(h)}(y) - S^{(h)}(y-v)}\right), & 0 \leq v \leq x - \beta + u. \end{cases}$$

Using (3.7) we compute the integrals above as follows

$$\int^z \frac{S^{(h)}(dy)}{S^{(h)}(y) - S^{(h)}(y-v)}$$
$$= \int^z \frac{(S(y-v) - S(\beta-u))\, S(dy)}{(S(y) - S(\beta-u))\,(S(y) - S(y-v))}$$
$$= -\ln\big(S(z) - S(\beta-u)\big) + \int^z \frac{S(dy)}{S(y) - S(y-v)}.$$

The claimed formula (3.1) results now from (3.8) when multiplied with the probability given in (3.6). ☐

Example 3.2. For BM(μ), Brownian motion with drift μ, the scale function of BM(μ) is

$$S(x) = \frac{1}{2\mu}\left(1 - \mathrm{e}^{-2\mu x}\right) =: f^{(\mu)}(x),$$

and it holds

$$\lim_{\mu\to 0} f^{(\mu)}(x) = x. \tag{3.9}$$

Applying Proposition 3.1 we write down in case $\mu > 0$ the joint distribution function $D^+_{H_\beta}$ and $D^-_{H_\beta}$. The integrals in (3.1) can be calculated explicitly and the result coincides with the one in Corollary 2.2 in [19]: for $\mu > 0$, $u > \beta$ and $x = 0$

$$\mathbf{P}_0^{(\mu)}(D^+_{H_\beta} < u\,;\, D^-_{H_\beta} < v) \tag{3.10}$$
$$= \begin{cases} \frac{f^{(\mu)}(u-\beta)}{f^{(\mu)}(u)}, & u \le v. \\ \frac{f^{(\mu)}(u-\beta)}{f^{(\mu)}(v)} \exp\left(-\frac{u-v}{f^{(-\mu)}(v)}\right), & u-\beta \le v \le u, \\ \exp\left(-\frac{\beta}{f^{(-\mu)}(v)}\right), & v \le u-\beta. \end{cases}$$

Letting here $\mu \to 0$ and using (3.9) we obtain the corresponding distribution of Brownian motion (without drift).

Example 3.3. Here we consider a 3-dimensional Bessel process with drift $\mu > 0$, for short BES($3,\mu$). This is a linear diffusion with the generator

$$\mathcal{G} = \frac{1}{2}\frac{d^2}{dx^2} + \mu \coth(\mu x)\frac{d}{dx}, \quad x > 0. \tag{3.11}$$

The scale function of BES$(3,\mu)$ is

$$S(x) = -2\mu\left(e^{2\mu x} - 1\right)^{-1} =: g^{(\mu)}(x), \quad x > 0. \tag{3.12}$$

Clearly,

$$\lim_{\mu \downarrow 0} g^{(\mu)}(x) = -\frac{1}{x},$$

which is the scale function of a 3-dimensional Bessel process (without drift). Notice also the connection

$$g^{(\mu)}(x) = -\frac{1}{f^{(-\mu)}(x)}$$

between the scale functions of Brownian motion with drift and 3-dimensional Bessel process with drift. Our first objective is to study the joint distribution of $D^-_{H_\beta}$ and $D^+_{H_\beta}$ when the process is initiated at $x > 0$. Notice that if $x = 0$ then $D^+_{H_\beta} = \beta$ a.s. and the joint distribution is singular. Secondly, we deduce the formula for a 3-dimensional Bessel process (without drift) by taking the limit as $\mu \downarrow 0$. We remark that in Proposition 2.5 in [19] the distribution of $D^-_{H_\beta}$ has already been given. From Proposition 3.1 it is seen that we need to evaluate the integrals appearing in (3.1): for $0 \leq x < \beta$

$$\int_x^\beta \frac{g^{(\mu)}(dy)}{g^{(\mu)}(y) - g^{(\mu)}(y-v)} = (2\mu - g^{(\mu)}(v))(\beta - x) + \log\left(\frac{g^{(\mu)}(\beta)}{g^{(\mu)}(x)}\right).$$

Moreover,

$$\frac{g^{(\mu)}(x) - g^{(\mu)}(\beta - u)}{g^{(\mu)}(\beta) - g^{(\mu)}(\beta - u)} = \frac{g^{(\mu)}(x) g^{(\mu)}(u)}{g^{(\mu)}(\beta) g^{(\mu)}(x - \beta + u)}$$

and

$$\frac{g^{(\mu)}(x) - g^{(\mu)}(\beta - u)}{g^{(\mu)}(v + \beta - u) - g^{(\mu)}(\beta - u)} = \frac{g^{(\mu)}(x) g^{(\mu)}(v)}{g^{(\mu)}(x - \beta + u) g^{(\mu)}(v + \beta - u)}.$$

These identities are convenient when writing down the explicit expression of the joint distribution of $D^-_{H_\beta}$ and $D^+_{H_\beta}$ for BES(3,μ) and also when taking the limits as $\mu \downarrow 0$. We focus now on the formula for 3-dimensional Bessel process (without drift). To this end, we have in case $0 \le \beta - x \le u \le \beta$ and $0 \le v \le \beta$

$$\mathbf{P}_x^{(Bes3)}(D^+_{H_\beta} < u\,;\, D^-_{H_\beta} < v) = \begin{cases} \frac{\beta(x-\beta+u)}{ux}, & u \le v. \\ \frac{\beta(x-\beta+u)}{vx}\exp\left(-\frac{u-v}{v}\right), & x-\beta+u \le v \le u, \\ \frac{\beta}{x}\exp\left(-\frac{\beta-x}{v}\right), & 0 \le v \le x-\beta+u. \end{cases}$$

In case $u \ge \beta$ and $v \ge 0$ it holds

$$\mathbf{P}_x^{(Bes3)}(D^-_{H_\beta} < v\,;\, D^+_{H_\beta} < u) = \mathbf{P}_x(D^-_{H_\beta} < v) = \begin{cases} 1, & v \ge \beta, \\ \frac{\beta}{v}\exp\left(-\frac{\beta-v}{v}\right), & x \le v \le \beta, \\ \frac{\beta}{x}\exp\left(-\frac{\beta-x}{v}\right), & 0 \le v \le x, \end{cases}$$

where we have applied also Proposition 2.3. Finally, in case $v \ge \beta$ and $u \ge 0$

$$\mathbf{P}_x^{(Bes3)}(D^-_{H_\beta} < v\,;\, D^+_{H_\beta} < u) = \mathbf{P}_x^{(Bes3)}(D^+_{H_\beta} < u) = \begin{cases} 1, & u \ge \beta, \\ \frac{\beta}{x}\frac{u-\beta+x}{u}, & \beta - x \le u \le \beta, \\ 0, & 0 \le u \le \beta - x, \end{cases}$$

where Proposition 2.2 is used.

Example 3.4. In this example we wish to draw the attention to the fact that, since the joint distribution of $D^-_{H_\beta}$ and $D^+_{H_\beta}$ only depend on the scale function, many different diffusions share the same distribution. This is, for instance, the case with the Bessel diffusion with index ν and the geometric

Brownian motion with volatility $\sigma^2 = 1$ and drift $\mu = \nu + 1/2$. The scale functions of these are given by (see [2] p. 137 and p.136)

$$S^{(Bes)}(x) = S^{(gbm)}(x) = \begin{cases} -\dfrac{x^{-2\nu}}{2\nu}, & \nu \neq 0 \\ \log x, & \nu = 0. \end{cases}$$

For $\nu \geq 0$ we have $S^{(Bes)}(0) = S^{(gbm)}(0) = -\infty$ and both processes converges a.s. to $+\infty$ as the time parameter tends to infinity. Proposition 3.1 can be applied to find the joint distribution of $D^-_{H_\beta}$ and $D^+_{H_\beta}$. However, the integral terms in (3.1) do not seem to allow explicit evaluations (for all values of ν).

4 Markov generator of D^-_H

We assume in this section for simplicity that $X(0) = 0 \in (l, r)$. From the strong Markov property of X it follows that $D^-_H = (D^-_{H_\beta})_{\beta \geq 0}$ is a Markov process with respect to the filtration $(\mathcal{F}_{H_\beta})_{\beta \geq 0}$. More precisely, since $D^-_{H_\alpha} \geq D^-_{H_\beta}$ for $\alpha > \beta \geq 0$, we have

$$\mathbf{P}_0 \left(D^-_{H_\alpha} > v \,|\, \mathcal{F}_{H_\beta} \right) = \begin{cases} 1, & v \leq D^-_{H_\beta}, \\ \mathbf{P}_\beta (D^-_{H_\alpha} > v), & v > D^-_{H_\beta}. \end{cases} \tag{4.1}$$

For a given $\beta > 0$ let

$$T_\beta := \inf\{\tau > \beta \,:\, D^-_{H_\tau} > D^-_{H_\beta}\},$$

i.e., T_β is the first jump "time" for the process D^-_H after "time" β. Then for $\gamma > \beta$

$$\mathbf{P}_0 \left(T_\beta > \gamma \,|\, \mathcal{F}_{H_\beta} \right) = \mathbf{P}_0 \left(D^-_{H_\gamma} = D^-_{H_\beta} \,|\, \mathcal{F}_{H_\beta} \right) = G(D^-_{H_\beta}), \tag{4.2}$$

where $G(u) := \mathbf{P}_\beta (D^-_{H_\gamma} < u)$. Next we calculate the generator of D^-_H.

Proposition 4.1. *Assume that the scale function S is in C^1. For $f \in C^1$ with compact support and for $\beta > 0$ it holds*

$$\lim_{\alpha \downarrow \beta} \frac{\mathbf{E}_0 (f(D^-_{H_\alpha}) \,|\, \mathcal{F}_{H_\beta}) - f(D^-_{H_\beta})}{\alpha - \beta} = \mathcal{A}_\beta f(D^-_{H_\beta}), \tag{4.3}$$

where $\mathcal{A}_\beta$ is the generator given by

$$\mathcal{A}_\beta f(u) := \int_0^{\beta-l-u} (f(z+u)-f(u))\,\frac{S'(\beta)S'(\beta-z-u)}{(S(\beta)-S(\beta-z-u))^2}\,dz.$$

Proof. We apply the strong Markov property of X to obtain for $\alpha > \beta$

$$D^-_{H_\alpha} \stackrel{(d)}{=} D^-_{H_\beta} \vee \widehat{D},$$

where, under $\mathbf{P}_0$, $D^-_{H_\beta}$ and $\widehat{D}$ are independent and $\widehat{D}$ is distributed as $D^-_{H_\alpha}$ under $\mathbf{P}_\beta$. Consequently,

$$\mathbf{E}_0(f(D^-_{H_\alpha})\,|\,\mathcal{F}_{H_\beta}) = Q\left(D^-_{H_\beta}\right),$$

where

$$\begin{aligned} Q(u) :&= \mathbf{E}_\beta(f(u \vee D^-_{H_\alpha})) \\ &= f(u)\mathbf{P}_\beta(D^-_{H_\alpha} \le u) + \mathbf{E}_\beta(f(D^-_{H_\alpha})\,;\ D^-_{H_\alpha} > u) \\ &= f(u) + \mathbf{E}_\beta\left((f(D^-_{H_\alpha}) - f(u))\,;\ D^-_{H_\alpha} > u\right) \\ &= f(u) + \int_u^\infty f'(y)\mathbf{P}_\beta(D^-_{H_\alpha} > y)\,dy. \end{aligned}$$

Hence,

$$\mathbf{E}_0(f(D^-_{H_\alpha})\,|\,\mathcal{F}_{H_\beta}) - f(D^-_{H_\beta}) = \int_{D^-_{H_\beta}}^\infty f'(y)\mathbf{P}_\beta(D^-_{H_\alpha} > y)\,dy. \tag{4.4}$$

We consider first the case $l = -\infty$. By (2.3) in Proposition 2.3

$$\mathbf{P}_\beta(D^-_{H_\alpha} > y) = 1 - \exp\left(-\int_\beta^\alpha \frac{S'(z)\,dz}{S(z)-S(z-y)}\right). \tag{4.5}$$

and, consequently,

$$\lim_{\alpha\downarrow\beta} \frac{\mathbf{P}_\beta(D^-_{H_\alpha} > y)}{\alpha-\beta} = \frac{S'(\beta)}{S(\beta)-S(\beta-y)}. \tag{4.6}$$

Applying dominated convergence which is justified since f is assumed to have a compact support we obtain now from (4.4) evoking (4.6)

$$\lim_{\alpha\downarrow\beta} \frac{\mathbf{E}_0(f(D^-_{H_\alpha})\,|\,\mathcal{F}_{H_\beta}) - f(D^-_{H_\beta})}{\alpha-\beta} = S'(\beta)\int_{D^-_{H_\beta}}^\infty \frac{f'(y)}{S(\beta)-S(\beta-y)}\,dy. \tag{4.7}$$

Finally integrating by parts on the right hand side of (4.7) and recalling that f has a compact support yield the claimed formula (4.3) after substituting $z = y - u$.

Assume next that $l > -\infty$. Notice that in this case $D^-_{H_\beta} < \beta - l$. If $0 \leq y \leq \beta - l$ formula (4.5) is still valid and results to (4.3). Hence, it remains to show that if $\beta - l < y < \alpha - l$ then the contribution is zero. From (2.3) we have

$$\mathbf{P}_\beta(D^-_{H_\alpha} > y) = 1 - \exp\left(-\int_{y+l}^{\alpha} \frac{S'(z)\,dz}{S(z) - S(z-y)}\right), \tag{4.8}$$

and straightforward calculations show that

$$\lim_{\alpha \downarrow \beta} \frac{1}{\alpha - \beta} \int_{\beta - l}^{\alpha - l} \mathbf{1}_{\{y > D^-_{H_\beta}\}} f'(y) \mathbf{P}_\beta(D^-_{H_\alpha} > y)\,dy = 0$$

because the right hand side of (4.8) equals zero when $y = \alpha - l$ and $\beta = \alpha$. This completes the proof. □

Remark 4.2. *D^-_H is a non-decreasing pure jump process having a finite number of jumps on any compact interval of $(0, +\infty)$. Its jump measure is given for any $u > 0$ by*

$$\nu_{u,\beta}(dz) = \frac{S'(\beta)S'(\beta - z - u)}{(S(\beta) - S(\beta - z - u))^2}\,dz, \qquad 0 < z < \beta - l - u. \tag{4.9}$$

We remark that $\nu_{u,\beta}$ is a finite measure with the total mass

$$\nu_{u,\beta}((0, +\infty)) = \frac{S'(\beta)}{S(\beta) - S(\beta - u)}, \quad \beta - u > l.$$

Remark 4.3. *If the left end point l is an entrance point the above analysis is still valid when $X(0) = l$. This is, for instance, the case with a 3-dimensional Bessel process where $l = 0$.*

Example 4.4. The jump measure for Brownian motion with drift $\mu \geq 0$ is for all positive u, β, and z given by

$$\nu^{(\mu)}_{u,\beta}(dz) = \begin{cases} (z+u)^{-2}\,dz, & \mu = 0, \\ \dfrac{4\mu^2 \mathrm{e}^{2\mu(z+u)}}{(\mathrm{e}^{2\mu(z+u)} - 1)^2}\,dz, & \mu > 0. \end{cases}$$

For 3-dimensional Bessel process with drift $\mu \geq 0$ the measure is as above but in this case $z \in (0, \beta - u)$ and $0 < u < \beta$. Notice that $\nu^{(\mu)} \to \nu^{(0)}$ as $\mu \downarrow 0$.

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