Finding an LFT uncertainty model with minimal uncertainty

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Finding an LFT Uncertainty Model with Minimal Uncertainty

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Abstract— **In this paper, we present a procedure for finding the best LFT uncertainty model by minimizing the $\mathcal{H}$-infinity norm of the uncertainty set with respect to a nominal model subject to known input-output data. The main problem is how to express the data-matching constraints for convenient use in the optimization problem. For some uncertainty structures, they can readily be formulated as a set of linear matrix inequalities (LMIs), for some other structures, LMIs are obtained after certain transformations. There are also cases, when the constraints result in bilinear matrix inequalities (BMIs), which can be linearized to enable an efficient iterative solution. Essentially all LFT uncertainty structures are considered. An application to distillation modeling is included.**

I. Introduction

For robust control design, a model with information about the model uncertainty is very useful. A standard model type for this purpose is a model consisting of a linear nominal model augmented by an uncertainty description in the form of a linear fractional transformation (LFT).

Up till recently there was no clear consensus as to what constitutes a control-oriented uncertainty set [1]. According to [2], it is essential that the identification methods deliver optimal uncertainty sets, including a nominal model, rather than an uncertainty bound around a prefixed nominal model. Thus, the uncertainty set should be minimized over both the nominal model and the uncertainty bound, as done, e.g., in [3]. For an LFT type of uncertainty, it has recently been shown that the $\mathcal{H}_\infty$ norm of the uncertainty is a rigorous measure of the worst-case degradation of the stability margin for a system under feedback control irrespective of the particular type of uncertainty structure [4].

If experimental input-output data are known, the uncertainty set can be minimized and an optimal nominal model can be determined by matching the uncertainty description to input-output data. Solutions have already been developed for additive uncertainty [3] and output-multiplicative uncertainty [5], but a general procedure for arbitrary LFT uncertainty models has been lacking. The alternative approach of model matching, where an uncertainty set is minimized and an optimal model is determined to enclose a set of known models, has also been considered for additive uncertainty [6] as well as multiplicative types of uncertainty [7]. However, it has been mathematically proved that model matching cannot provide a smaller uncertainty bound than input-output matching when the same input-output data has to be satisfied [7].

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In this paper we present a procedure for finding the best LFT uncertainty model by minimizing the $\mathcal{H}_\infty$ norm of the uncertainty with respect to a nominal model subject to known input-output data. Essentially all LFT uncertainty structures are considered. An application to distillation modeling is included.

II. Problem Formulation

An uncertainty model expressed as a linear fractional transformation has the general form

$$G = G_0 + H_{21}\Delta(I - H_{11}\Delta)^{-1} H_{12}\,, \tag{1}$$

where G is a transfer function of the true system, G_0 is the transfer function of a nominal model, and Δ is the transfer function of an unknown bounded perturbation causing uncertainty about the true system. Depending on the particular type of uncertainty (e.g., additive, input or output multiplicative, inverse types of uncertainty, combinations of various types of uncertainty), the matrices H_{11}, H_{12} and H_{21} are constant matrices or various combinations of constant matrices and the nominal transfer function. Well-posedness (stability) of the model requires $\|H_{11}\Delta\|_\infty < 1$. In general, the uncertainty Δ may be structured or unstructured, but in this paper we deal with unstructured uncertainty only.

Assuming (1) to be time-invariant with all matrices proper and real rational, it has been shown that

$$\|\Delta\|_\infty := \sup_\omega \bar{\sigma}\big(\Delta(\mathrm{j}\omega)\big) \tag{2}$$

is a rigorous measure of the worst-case degradation of the stability margin due to a perturbation Δ of a system (1) under feedback control [4]. In (2), $\bar{\sigma}(A)$ denotes the largest singular value of A and ω denotes frequency. Obviously, an uncertainty description having a smaller $\|\Delta\|_\infty$ than another uncertainty description is preferable from the viewpoint of feedback control. Hence, this can be used as a basis for choosing an uncertainty model for robust control design.

It is assumed that a number of identification experiments have been performed to give data for finding an optimal uncertainty model structure. Thus, input-output data $\{u_k, y_k\}$, $k = 1,\dots,N$, are assumed to be known for N experiments. These data satisfy

$$y_k = G_0 u_k + H_{21}\Delta_k (I - H_{11}\Delta_k)^{-1} H_{12} u_k\,, \tag{3}$$

where Δ_k is a model perturbation required to satisfy the data in experiment k. It is clear that the size of Δ_k depends not only on the type of uncertainty description (i.e., choice of

H_{11}, H_{12} and H_{21}), but also on the choice of G_0. For a given uncertainty description and an optimal choice of G_0, the size of the smallest uncertainty required to satisfy the data of all experiments is thus

$$\min \|\Delta\|_\infty = \min_{G_0} \max_k \|\Delta_k\|_\infty . \tag{4}$$

Because the optimization involves the $\mathcal{H}_\infty$ norm of the uncertainty, G_0 is not uniquely determined by (4). To find an optimal G_0, instead of solving (4), we minimize $\bar{\sigma}(\Delta(\mathrm{j}\omega))$ for a selection of frequency points, $\omega \in \Omega$. This gives sampled frequency responses of G_0 according to

$$G_0(\mathrm{j}\omega) = \arg \min_{G_0(\mathrm{j}\omega)} \max_k \bar{\sigma}(\Delta_k(\mathrm{j}\omega)),\ \forall \omega \in \Omega . \tag{5}$$

A nominal transfer function $\tilde{G}_0(s)$ can then be found by fitting $\tilde{G}_0(\mathrm{j}\omega)$ to $G_0(\mathrm{j}\omega)$ subject to the appropriate data matching constraints to ensure that the uncertainty model satisfies known input-output data [8]. The transfer function fitting is not treated in the present paper, however.

Since the optimization in (5) is done subject to (3), frequency samples of the input-output data have to be available. It is usually not a problem to calculate $u_k(\mathrm{j}\omega)$, $\omega \in \Omega$, for the input in a controlled identification experiment. A convenient way of obtaining $y_k(\mathrm{j}\omega)$ is to fit a filter, G_k, to the input-output data and to calculate a noise-free output according to $y_k(\mathrm{j}\omega) = G_k(\mathrm{j}\omega)u_k(\mathrm{j}\omega)$. Note that G_k is not assumed to be a true model appropriate for model matching, because u_k might not be persistently exciting in the various experiments [9].

The main problem treated in this paper is how to express the data-matching constraint (3) for convenient use in the optimization problem. For some uncertainty structures, (3) can readily be formulated as a set of linear matrix inequalities (LMIs), for some other uncertainty structures, LMIs are obtained after certain transformations of (3). There are also cases, when the constraints result in bilinear matrix inequalities (BMIs), which can be linearized to enable an efficient iterative solution. Essentially all known uncertainty structures of LFT type are considered. This makes it possible to find the uncertainty structure having the smallest $\|\Delta\|_\infty$ for the given sets of data.

III. LFT Uncertainty Model Structures

A. Left 4-Block Interconnection

Fig. 1 illustrates a so-called left 4-block interconnection of an uncertainty structure [4]. It is defined by

$$y = G_0(u - w_2) + w_1 , \tag{6}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = S_w \Delta S_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{7}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix}, \tag{8}$$

where z_1 and z_2 are perturbation signals from the output side and the input side, respectively; w_1 and w_2 are disturbances affecting the output and the input, respectively; S_z and S_w are perturbation and disturbance selector matrices.

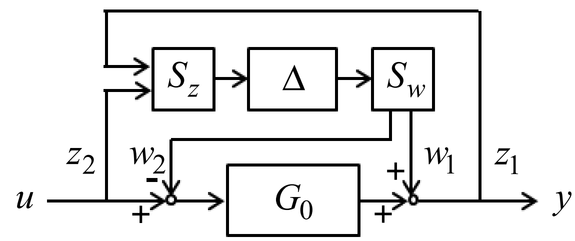

Fig. 1. Left 4-block interconnection.

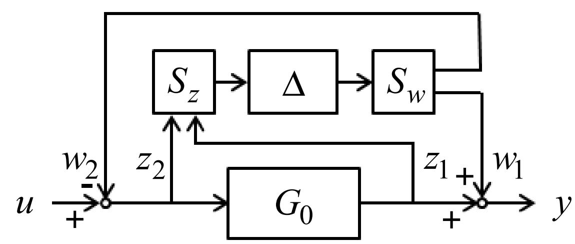

Fig. 2. Right 4-block interconnection.

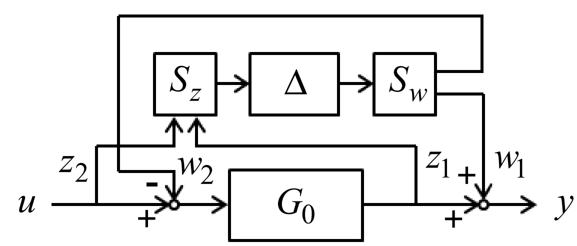

Fig. 3. A mixed 4-block interconnection.

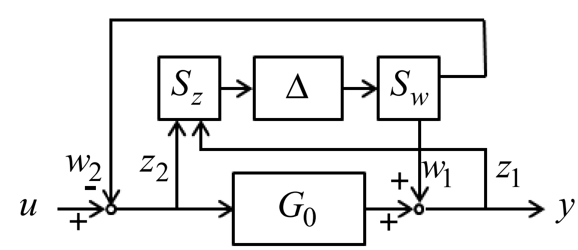

Fig. 4. Another mixed 4-block interconnection.

The selector matrices define the particular uncertainty structure within the left 4-block interconnection structure. The choices

$$S_z \in \left\{ \begin{bmatrix} I & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right\},\ S_w \in \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right\} , \tag{9}$$

give rise to various uncertainty structures such as additive, input multiplicative, inverse additive and output inverse multiplicative uncertainty as well as combinations of these.

The special choice [4]

$$S_z = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \ S_w = \begin{bmatrix} \tilde{M}_0^{-1} \\ 0 \end{bmatrix}, \tag{10}$$

where $\tilde{M}_0$ is a left-coprime factor of G_0, gives a left-coprime factorized uncertainty model.

B. Other 4-Block Interconnections

There are also other LFT interconnections. Fig. 2 illustrates a right 4-block interconnection defined by (6), (7) and

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{11}$$

With the chosen terminology, Fig. 3 and 4 illustrate mixed types of LFT interconnections. In addition to (6) and (7), the interconnection in Fig. 3 is defined by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{12}$$

and the interconnection in Fig. 4 by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{13}$$

In general, the interconnections are defined by (6), (7) and

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix} - S\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \ S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}, \tag{14}$$

where $S_{11}, S_{22} \in \{0, I\}$.

C. Model for All Interconnections

Combination of (7) and (14) gives

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = S_w\Delta\left(I + S_z S S_w \Delta\right)^{-1} S_z \begin{bmatrix} y \\ u \end{bmatrix}, \tag{15}$$

which inserted into (6) yields the "mixed" input-output model

$$y = G_0 u + \begin{bmatrix} I & -G_0 \end{bmatrix} S_w \Delta\left(I + S_\Delta \Delta\right)^{-1} S_z \begin{bmatrix} y \\ u \end{bmatrix}, \tag{16}$$

where

$$S_\Delta := S_z S S_w. \tag{17}$$

Solving (16) for y gives the uncertainty model

$$y = G_0 u + \begin{bmatrix} I & -G_0 \end{bmatrix} S_w \Delta\left(I + \tilde{S}_\Delta \Delta\right)^{-1} S_z \begin{bmatrix} G_0 \\ I \end{bmatrix} u, \tag{18}$$

where

$$\tilde{S}_\Delta := S_z \tilde{S} S_w, \ \tilde{S} := S - \begin{bmatrix} I \\ 0 \end{bmatrix}\begin{bmatrix} I & -G_0 \end{bmatrix}. \tag{19}$$

We note that the matrices in (1) and (3) are

$$H_{11} = -\tilde{S}_\Delta, \ H_{12} = S_z \begin{bmatrix} G_0 \\ I \end{bmatrix}, \ H_{21} = \begin{bmatrix} I & -G_0 \end{bmatrix} S_w. \tag{20}$$

IV. Deriving Data-Matching Constraints

A. Problem Overview

Depending on S_z, S_w and S, there are various problems that have to be solved in the derivation of data-matching constraints for the optimization in (5).

- If $\tilde{S}_\Delta \neq 0$, Δ does not appear linearly in (18). If $S_\Delta = 0$, (16) can be used instead of (18), but otherwise, a linearizing transformation has to be done.
- If $S_z \neq \begin{bmatrix} 0 & I \end{bmatrix}$, use of (18) will introduce a bilinear term containing $G_0^* G_0$ in the data matching constraint. This can be avoided by using (16).
- If $S_w = I$, both (16) and (18) will introduce a bilinear term containing $G_0 G_0^*$, which gives a BMI problem.
- If $S_w = \begin{bmatrix} 0 & I \end{bmatrix}^{\mathrm{T}}$, the BMI problem can be avoided by multiplying the terms in (16) or (18) by G_0^{-1} from the left (assuming G_0 to be invertible).

From (18) it follows that well-posedness of the uncertainty structure requires $\|\Delta\|_\infty < \|\tilde{S}_\Delta\|_\infty^{-1}$. When (16) is used instead of (18), the well-posedness restriction becomes $\|\Delta\|_\infty < \|S_\Delta\|_\infty^{-1}$. For many choices of S_z, S_w and S, we have $S_\Delta = I$. This gives the requirement $\|\Delta\|_\infty < 1$, which might be tighter than the true well-posedness requirement of (18). On the other hand, in many cases the well-posedness restriction vanishes when (16) is used, which makes it easier to find a numerical solution.

B. Two Useful Lemmas

We shall use two lemmas adapted from [10].

Lemma 1. Consider the matrix equation

$$C = A\Delta B. \tag{21}$$

There is a solution for Δ, $\bar{\sigma}(\Delta) \leq 1$, if and only if

$$\begin{bmatrix} AA^* & C \\ C^* & B^* B \end{bmatrix} \succcurlyeq 0. \tag{22}$$

Here, X^* denotes the complex-conjugate transpose of X and $P \succcurlyeq 0$ denotes that P is a positive semidefinite matrix.

Lemma 2. Consider the matrix equation

$$G = H_{22} + H_{21}\Delta(1 - H_{11}\Delta)^{-1} H_{12}, \tag{23}$$

where $\bar{\sigma}(H_{11}) < 1$ and $\bar{\sigma}(\Delta) \leq 1$. For every Δ, there is a $\hat{\Delta}$ satisfying

$$G = \hat{H}_{22} + \hat{H}_{21}\hat{\Delta}\hat{H}_{12}, \tag{24}$$

where

$$\begin{aligned} \hat{H}_{22} &:= H_{22} + H_{21}H_{11}^*(I - H_{11}H_{11}^*)^{-1} H_{12} \\ \hat{H}_{21} &:= -H_{21}(I - H_{11}^* H_{11})^{-1/2} \\ \hat{H}_{12} &:= (I - H_{11}H_{11}^*)^{-1/2} H_{12} \end{aligned} \tag{25}$$

if and only if $\bar{\sigma}(\hat{\Delta}) \leq 1$. For every $\hat{\Delta}$, $\bar{\sigma}(\hat{\Delta}) \leq 1$, there is also a Δ satisfying (23) if and only if $\bar{\sigma}(\Delta) \leq 1$.

In the lemmas, $\bar{\sigma}(\Delta) \leq 1$ is required. For an uncertainty $\bar{\sigma}(\Delta) \leq \delta$, we can formally impose this condition by writing the uncertainty as $\delta\Delta$, $\bar{\sigma}(\Delta) \leq 1$, and include the real-valued scalar δ in a matrix adjacent to Δ in (21) (i.e., A or B) and (23) (i.e., H_{11} and H_{21}).

V. Main Results

In this section, data-matching constraints are derived for all LFT uncertainty models, represented by (16) and/or (18). The constraints are expressed in terms of input-output data $\{u_k, y_k\}$, $k = 1, \ldots, N$, and the perturbations are expressed as $\delta_k \Delta_k$, $\bar{\sigma}(\Delta_k) \leq 1$. For convenience, the combined vector of outputs and inputs is denoted

$$x_k := \begin{bmatrix} y_k \\ u_k \end{bmatrix}. \tag{26}$$

The expressions apply individually for every considered frequency, $\omega \in \Omega$, but the frequency argument '$\mathrm{j}\omega$' is not included.

A. $S_w = [I \ \ 0]^{\mathrm{T}}$ and $S_\Delta = 0$

In this case, there is no input disturbance (the S_w condition) and no connection between a disturbance and a possible perturbation on the output side ($S_z = [0 \ \ I]$ or $S_{11} = 0$, resulting in $S_\Delta = 0$).

The mixed input-output expression (16) yields

$$y_k = G_0 u_k + \delta_k \Delta_k S_z x_k. \tag{27}$$

Application of Lemma 1 now gives the condition

$$\begin{bmatrix} \alpha I & y_k - G_0 u_k \\ (*) & x_k^* S_z^* S_z x_k \end{bmatrix} \succcurlyeq 0, \tag{28}$$

where $(*)$ indicates the complex-conjugate transpose of the elements in the symmetrical position of the full matrix. Here,

$$\alpha = \max_k \delta_k^2. \tag{29}$$

The solution of the optimization problem simplifies to

$$G_0(\mathrm{j}\omega) = \arg \min_{G_0(\mathrm{j}\omega)} \alpha \quad \text{s.t. (28)}, \ \forall k. \tag{30}$$

Note that the problem is well-posed for arbitrarily large uncertainties (δ_k) even though (18) restricts the allowable uncertainty by $\|\Delta\|_\infty \leq 1$ if $S_z \neq [0 \ \ I]$.

B. $S_w = [I \ \ 0]^{\mathrm{T}}$, $S_z \neq [0 \ \ I]$, and $\tilde{S}_\Delta = 0$

This case occurs when there is no input disturbance (S_w condition), but a connection between a disturbance and a perturbation on the output side (S_z condition and $S_{11} = I$, resulting in $\tilde{S}_\Delta = 0$).

The pure input-output expression (18) gives

$$y_k = G_0 u_k + \delta_k \Delta_k S_z \begin{bmatrix} G_0 \\ I \end{bmatrix} u_k \tag{31}$$

for which Lemma 1 yields

$$\begin{bmatrix} \alpha I & y_k - G_0 u_k \\ (*) & u_k^* G_0^* G_0 u_k + u_k^* S_{zu} u_k \end{bmatrix} \succcurlyeq 0, \tag{32}$$

where $S_{zu} = 0$ if $S_z = [I \ \ 0]$, $S_{zu} = I$ if $S_z = I$.

Unfortunately, (32) is a BMI when G_0 is a decision variable. Using (16) instead of (18) gives

$$y_k = G_0 u_k + \delta_k \Delta_k \left(I + S_z S_w \delta_k \Delta_k\right)^{-1} S_z x_k, \tag{33}$$

which is nonlinear in terms of Δ_k. However, by application of Lemma 2, (33) can be reformulated as

$$y_k = (1 - \delta_k^2) G_0 u_k - \delta_k \hat{\Delta}_k S_z \begin{bmatrix} y_k \\ (1 - \delta_k^2)^{1/2} u_k \end{bmatrix}, \tag{34}$$

which by use of Lemma 1 gives

$$\begin{bmatrix} \beta I & (1+\beta) y_k - G_0 u_k \\ (*) & (1+\beta) y_k^* y_k + u_k^* S_{zu} u_k \end{bmatrix} \succcurlyeq 0, \tag{35}$$

where

$$\beta = \max_k \delta_k^2 (1 - \delta_k^2)^{-1} \geq 0. \tag{36}$$

The solution of the optimization problem is given by

$$G_0(\mathrm{j}\omega) = \arg \min_{G_0(\mathrm{j}\omega)} \beta \quad \text{s.t. (35)}, \ \forall k. \tag{37}$$

Note that the use of (16) and Lemma 2 in this case introduces the restriction $0 \leq \delta_k < 1$ although the original formulation in (18) does not have such a restriction. If this restriction is unacceptable, the optimization must be based on (32) and solved as described in Subsection *E*.

C. $S_w = [0 \ \ I]^{\mathrm{T}}$ and $S_\Delta = 0$

This case differs from Case A in that there is an input disturbance, but no output disturbance (S_w condition), and no connection between a disturbance and a possible perturbation on the input side ($S_z = [I \ \ 0]$ or $S_{22} = 0$, resulting in $S_\Delta = 0$).

The mixed input-output expression (16) yields

$$y_k = G_0 u_k - G_0 \delta_k \Delta_k S_z x_k. \tag{38}$$

Direct application of Lemma 1 would result in a BMI, but by first multiplying every term in (38) by G_0^{-1} from the left (assuming G_0 to be invertible), we obtain

$$\begin{bmatrix} \alpha I & G_0^{-1} y_k - u_k \\ (*) & x_k^* S_z^* S_z x_k \end{bmatrix} \succcurlyeq 0 \tag{39}$$

with the optimization problem

$$G_0^{-1}(\mathrm{j}\omega) = \arg \min_{G_0^{-1}(\mathrm{j}\omega)} \alpha \text{ s.t. (39)}, \ \forall k . \tag{40}$$

D. $S_w = \begin{bmatrix} 0 & I \end{bmatrix}^{\mathrm{T}}$ *and* $S_\Delta = I$ *or* $\begin{bmatrix} 0 & I \end{bmatrix}^{\mathrm{T}}$

In this case, there is no output disturbance (S_w condition), but a connection between a disturbance and a perturbation on the input side ($S_z \neq \begin{bmatrix} I & 0 \end{bmatrix}$ and $S_{22} = I$, resulting in the S_Δ condition).

Both the mixed and pure input-output relations are now nonlinear with respect to Δ_k. However, they can be linearized by a simple reformulation. Pre-multiplication by G_0^{-1}, use of the identity $A(I+BA)^{-1} = (I+AB)^{-1}A$, and some further manipulations, result in

$$G_0^{-1} y_k = u_k - \delta_k \Delta_k S_z \begin{bmatrix} I \\ G_0^{-1} \end{bmatrix} y_k . \tag{41}$$

Lemma 1 now gives

$$\begin{bmatrix} \alpha I & G_0^{-1} y_k - u_k \\ (*) & y_k^* G_0^{-*} G_0^{-1} y_k + y_k^* S_{zy} y_k \end{bmatrix} \succcurlyeq 0 , \tag{42}$$

where $S_{zy} = 0$ if $S_z = \begin{bmatrix} 0 & I \end{bmatrix}$, $S_{zy} = I$ if $S_z = I$.

Equation (42) is a BMI with respect to the decision variable G_0^{-1}. By application of Lemma 2 to the mixed input-output relation (16), we can similarly as in Subsection *B* derive

$$(1-\delta_k^2) G_0^{-1} y_k = u_k - \delta_k \hat{\Delta}_k S_z \begin{bmatrix} (1-\delta_k^2)^{1/2} y_k \\ u_k \end{bmatrix} . \tag{43}$$

By Lemma 1 we then obtain

$$\begin{bmatrix} \beta I & (1+\beta) u_k - G_0^{-1} y_k \\ (*) & (1+\beta) u_k^* u_k + y_k^* S_{zy} y_k \end{bmatrix} \succcurlyeq 0 . \tag{44}$$

Also here, use of Lemma 2 introduces the restriction $0 \le \delta_k < 1$. However, this restriction is not tighter than the well-posedness restriction $\bar{\sigma}(\Delta_k)\bar{\sigma}(\tilde{S}_\Delta) < 1$ in the pure input-output relation (18). If the optimization is done subject to the BMI (42), an unrestricted solution can be obtained.

E. $S_w = I$ *and* $S_\Delta = 0$ *or* I

In this case, there are input as well as output disturbances (S_w condition). The condition $S_\Delta = 0$ means that there is no direct connection between perturbations and disturbances, whereas $S_\Delta = I$ means that there are connections both on the input and the output side. It turns out that both cases result in the same type of data-matching constraints.

Because $S_w = I$, it is clear that BMIs cannot be avoided by the techniques presented above. For $S_\Delta = 0$, (16) yields

$$y_k = G_0 u_k + \begin{bmatrix} I & -G_0 \end{bmatrix} \delta_k \Delta_k S_z x_k , \tag{45}$$

for which Lemma 1 gives

$$\begin{bmatrix} I + G_0 G_0^* & y_k - G_0 u_k \\ (*) & \alpha x_k^* S_z^* S_z x_k \end{bmatrix} \succcurlyeq 0 . \tag{46}$$

For $S_\Delta = I$, application of Lemma 2 to (16) gives the same expression as (45) with Δ_k replaced by $\hat{\Delta}_k$. Thus, the data-matching constraint (46) is also the same.

A left-coprime factorized (LCF) uncertainty model is obtained with $S = 0$ and (10). This will also result in the mixed input-output model (45). Thus, a LCF model is equivalent with a full right 4-block interconnection with $S_z = S_w = S = I$.

The bilinearity of (46) is caused by the term $G_0 G_0^*$. Assume now that a suboptimal solution $\bar{G}_0$ has been found. Linearization of $G_0 G_0^*$ around this solution gives

$$\begin{aligned} G_0 G_0^* &\approx \bar{G}_0 \bar{G}_0^* + \bar{G}_0 (G_0^* - \bar{G}_0^*) + (G_0 - \bar{G}_0) \bar{G}_0^* \\ &= \bar{G}_0 G_0^* + G_0 \bar{G}_0^* - \bar{G}_0 \bar{G}_0^* . \end{aligned} \tag{47}$$

Substitution of this approximation into (46) gives

$$\begin{bmatrix} I - \bar{G}_0 \bar{G}_0^* + \bar{G}_0 G_0^* + G_0 \bar{G}_0^* & y_k - G_0 u_k \\ (*) & \alpha x_k^* S_z^* S_z x_k \end{bmatrix} \succcurlyeq 0 , \tag{48}$$

which is now linear in the decision variable G_0.

Because the second order perturbation neglected in the linearization is a positive (semi)definite matrix,

$$G_0 G_0^* \succcurlyeq \bar{G}_0 G_0^* + G_0 \bar{G}_0^* - \bar{G}_0 \bar{G}_0^* . \tag{49}$$

This means that (48) is a tighter constraint than the exact constraint (46) as long as the solution is suboptimal. This, in turn, means that the solution of an iteration using (48) will always satisfy (46). Thus, there is a smaller α than the current solution $\alpha = \bar{\alpha}$ that will satisfy (46) tightly. Since (48) is equivalent with (46) when $G_0 = \bar{G}_0$, a smaller α will be found when the iteration is continued with (48). From this it follows that the solution will converge to the optimal solution satisfying (46).

F. $S_w = I$, $S_\Delta = \begin{bmatrix} 0 & I \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$

In this case, there are input and output disturbances (S_w condition) and a direct connection between a perturbation and a disturbance on the input side; there is no direct connection between a possible perturbation and disturbance on the output side (S_Δ condition).

The input-output expressions are here more complicated than in the previous cases. Equation (16) will give the data-matching condition

$$\begin{bmatrix} (1-\alpha)I + G_0 G_0^* & (1-\alpha)y_k - G_0 u_k \\ (*) & \alpha x_k^* S_z^* S_z x_k - \alpha^2 S_{zy} y_k^* y_k \end{bmatrix} \succcurlyeq 0 . \quad (50)$$

This expression is bilinear both with respect to G_0 and α. The bilinearity in α can be handled by the substitution $\alpha^2 = \alpha\bar{\alpha}$, where $\bar{\alpha}$ is kept constant during each iteration and updated before next iteration. When $\bar{\alpha}$ is larger than the optimal solution, the true bilinear constraint will always be satisfied when (50) is satisfied and α will converge to the optimal solution. Note that $0 \le \alpha < 1$ is required. The bilinearity $G_0 G_0^*$ can be handled in the same way as in Subsection *E*.

G. $S_w = I$, $S_\Delta = [I \ \ 0]$ *or* $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

Here, there are input and output disturbances (S_w condition) and a direct connection between a perturbation and a disturbance on the output side; there is no direct connection between a possible perturbation and disturbance on the input side (S_Δ condition).

In this case the data-matching condition becomes

$$\begin{bmatrix} (1+\beta)I + G_0 G_0^* & (1+\beta)y_k - G_0 u_k \\ (*) & \beta y_k^* y_k + \beta(1+\beta)^{-1} S_{zu} u_k^* u_k \end{bmatrix} \succcurlyeq 0 . \quad (51)$$

Also here, the expression is bilinear with respect to both G_0 and the optimization parameter β. The bilinearities can be handled similarly as in Subsection *F*. In this case, $(1+\beta)^{-1}$ is replaced by $(1+\bar{\beta})^{-1}$.

VI. Application to Distillation Modeling

In this section the presented uncertainty modeling method is applied to a two-product distillation column. Identification data determined in [9] are used for the modeling. The distillation column was excited by a series of step changes in the so-called high- and low-gain directions. This resulted in six sets of input-output data, $\{u_k, y_k\}$, $k = 1,\ldots,6$. A preliminary nominal model was determined by least-squares fitting to all available data. The same data have also been used in uncertainty modeling by other, less general, methods [3], [6] for some uncertainty structures.

A. Additive Uncertainty

Additive uncertainty ($S_z = [0 \ \ I]$, $S_w = [I \ \ 0]^{\mathrm{T}}$) is an example of a *Case A* uncertainty structure. Fig. 5 shows the minimum uncertainty norm as function of frequency for both the preliminary (original) nominal model and the optimal nominal model.

B. Inverse Output Multiplicative Uncertainty

Fig. 6 illustrates inverse output multiplicative uncertainty, which is obtained by $S_z = [I \ \ 0]$ and $S_w = [I \ \ 0]^{\mathrm{T}}$ in *Case A*. Now there is also a well-posedness limit $\bar{\sigma}(\Delta) \le 1$, which is included in the figure. As can be seen, the original nominal model does not satisfy this limit for all frequencies. Overall, the uncertainty even for the optimal model is much larger than for additive uncertainty.

C. Output Multiplicative Uncertainty

Output multiplicative uncertainty is obtained in *Case B* by $S_z = [I \ \ 0]$ and $S_w = [I \ \ 0]^{\mathrm{T}}$. Fig. 7 illustrates the result. It is very similar to inverse output multiplicative uncertainty, but in this case there is no well-posedness limit.

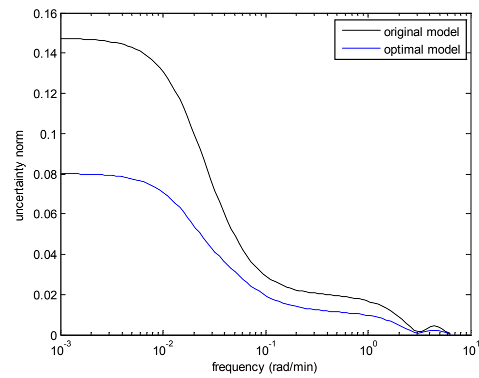

Fig. 5. Additive uncertainty.

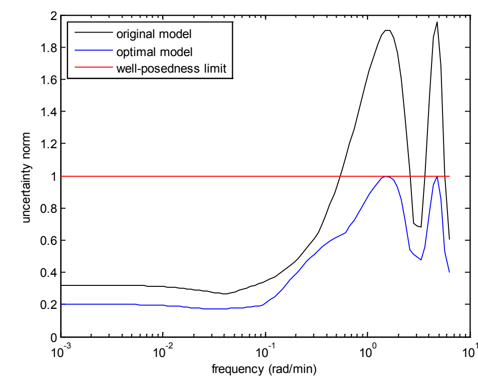

Fig. 6. Inverse output multiplicative uncertainty.

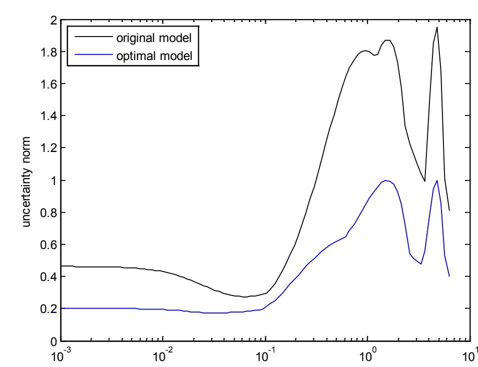

Fig. 7. Output multiplicative uncertainty.

D. *Input Multiplicative Uncertainty*

Input multiplicative uncertainty is obtained in *Case C* with $S_z = [0 \quad I]$ and $S_w = [0 \quad I]^{\mathrm{T}}$. As can be seen in Fig. 8, the original nominal model performs very badly, but also the optimal nominal model gives an uncertainty norm close to 1 for all frequencies. There is no well-posedness limit.

E. *Inverse Additive Uncertainty*

Inverse additive uncertainty is obtained from *Case C* with $S_z = [I \quad 0]$ and $S_w = [0 \quad I]^{\mathrm{T}}$. For this uncertainty structure, both the original and the optimal model perform very badly. There is also a well-posedness limit, which is not satisfied at some frequencies even by the optimal model. See Fig. 9.

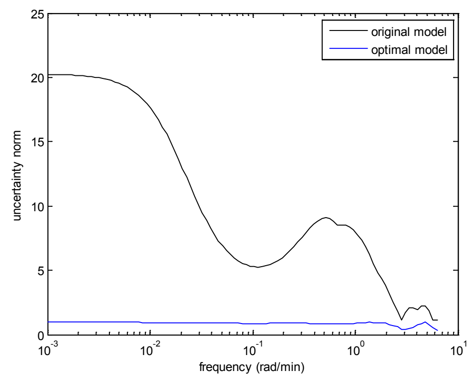

Fig. 8. Input multiplicative uncertainty.

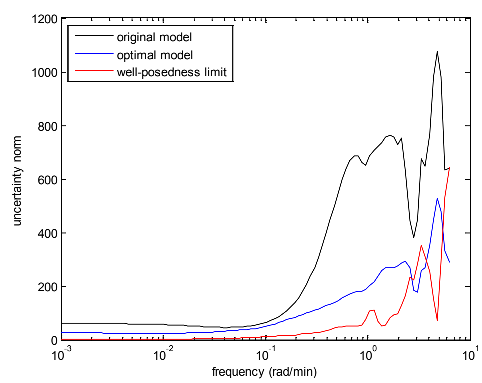

Fig. 9. Inverse additive uncertainty.

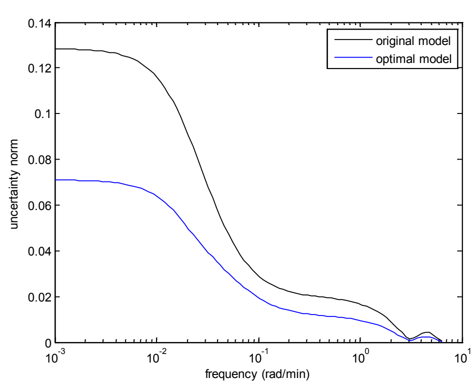

Fig. 10. Left-coprime factorized uncertainty.

F. *Left-Coprime Factorized Uncertainty*

Left-coprime factorized (LCF) uncertainty is obtained from *Case E* with $S_z = S_w = S = I$. As Fig. 10 shows, this uncertainty structure performs slightly better than additive uncertainty. However, the uncertainty block in LCF uncertainty is twice as large as in additive uncertainty.

G. *Other Uncertainty Structures*

All other uncertainty structures perform similarly to some of the already presented ones. However, for many of these, the dimension of the uncertainty block is larger than the basic size $\dim(y) \times \dim(u)$.

Considering the size of the uncertainty block, additive uncertainty is probably the best uncertainty structure for the data in this application.

VII. Conclusion

We have presented a procedure using convex optimization techniques for finding the tightest $\mathcal{H}_\infty$ bound on the unstructured uncertainty of any LFT uncertainty model such that it is not invalidated by known input-output data. By systematically considering all LFT models, we can find the best one (in terms of smallest $\mathcal{H}_\infty$ norm) for robust control design. In this paper, only the frequency response of the resulting nominal model was determined at a number of frequency points. A parametric model (e.g., a state-space model) can be obtained by fitting it to the frequency response subject to the known data-matching constraints.

References

[1] M. Gevers, "Identification for control: From the early achievements to the revival of experiment design," *European J. of Control*, vol. 11, no. 4–5, pp. 335–352, Dec. 2005.

[2] S. G. Douma and P. M. J. van den Hof, "Relations between uncertainty structures in identification for robust control," *Automatica*, vol. 41, no. 3, 439–457, March 2005.

[3] K. E. Häggblom, "Data-based uncertainty modeling by convex optimization techniques," in *Proc. Int. Symp. on Advanced Control of Chemical Processes* (ADCHEM), Gramado, Brazil, 2006, pp. 91–96.

[4] A. Lanzon and G. Papageorgiou, "Distance measures for uncertain linear systems: a general theory," *IEEE Trans. Automat. Control*, vol. 54, no. 7, pp. 1532–1547, July 2009.

[5] H. Hindi, C-Y. Seong, and S. Boyd, "Computing optimal uncertainty models from frequency domain data," in *Proc. 41st IEEE Conf. Decision and Control*, Las Vegas, NV USA, 2002, pp. 2898–2905.

[6] K. E. Häggblom, "Convex formulations for data-based uncertainty minimization of linear uncertainty models," in *Proc. 11th Int. Conf. Control, Automation, Robotics and Vision*, Singapore, 2010, pp. 501–505.

[7] R. H. Nyström, K. E. Häggblom, and J. M. Böling, "Derivation and selection of norm-bounded uncertainty descriptions based on multiple models," *Int. J. Control*, vol. 76, no. 7, pp. 717–727, May 2003.

[8] K. E. Häggblom, "MIMO uncertainty model identification of time-delay systems," in *Proc. 8th IFAC Symp. on Advanced Control of Chemical Processes* (ADCHEM), Singapore, 2012, pp. 385–390.

[9] K. E. Häggblom and J. M. Böling, "Multimodel identification for control of an ill-conditioned distillation column," *J. Process Control*, vol. 8, no. 3, pp. 209–218, June 1998.

[10] J. Chen and G. Gu, *Control-Oriented System Identification: An $\mathcal{H}_\infty$ Approach*, New York: Wiley-Interscience, 2000.