Identification of an LFT uncertainty model by minimizing the nu-gap metric

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Identification of an LFT Uncertainty Model by Minimizing the ν-Gap Metric*

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***Abstract*— An uncertainty model in the form of a linear fractional transformation (LFT) is composed of a nominal model augmented by an uncertainty description. The size of the uncertainty required to cover a given set of models depends not only on the set of models, but also on the nominal model. Thus, the size of the uncertainty can be minimized by choosing the nominal model optimally according to some metric. If the uncertainty model is to be used for control design, a suitable metric is the ν-gap metric. It is shown that the optimal solution in terms of the ν-gap metric has to satisfy a bilinear matrix inequality (BMI) for every model in the model set. To solve this non-convex optimization problem, the BMIs are linearized to enable an iterative solution constrained by linear matrix inequalities (LMIs), where each iteration is a convex optimization problem. It is proved that the iteration converges to the optimal solution satisfying the BMIs. Because the solution is obtained as the frequency response at selected frequencies, the final model is determined by fitting a model to the frequency responses. A state-space model is used because the fitting can then easily be done subject to the same BMIs/LMIs to guarantee an optimal model. The procedure is illustrated by an application to uncertainty modeling of the product composition dynamics of a distillation column.**

I. Introduction

It is well known that systems which are close to each other as measured by an operator norm may behave very differently under feedback control, and vice versa. To overcome this problem, the gap metric was introduced by Zames and El-Sakkary [1] and further elaborated by El-Sakkary [2]. It was shown that the robustness properties of feedback systems can be related to this metric. Georgiou and Smith [3] proved that robustness optimization in the gap metric is equivalent to robustness optimization for perturbations in the factors of a coprime-factorized model.

The gap metric, which is based on the small-gain theorem proposed by Zames [4], only allows perturbations that belong to $\mathcal{RH}_\infty$. Glover [5] has extended the small-gain theorem to allow perturbations in $\mathcal{RL}_\infty$ provided that a certain condition related to the right-half plane (RHP) pole/zero count holds. Based on this, Vinnicombe [6, 7] proposed a new measure, the ν-gap metric, which allows perturbations in $\mathcal{RL}_\infty$ of the factors of a normalized coprime-factorized model. For linear time-invariant systems, the ν-gap metric is easier to calculate than the gap metric and it gives less conservative results because of the perturbation classes.

The ν-gap metric has been applied to areas such as system identification [8–10], model/controller reduction [11, 12], and adaptive control employing multiple models [13].

In this paper, the ν-gap metric is applied to uncertainty modeling. A linear uncertainty model is composed of a nominal linear model augmented by an uncertainty description. A standard type of uncertainty model is one where the uncertainty description is in the form of a linear fractional transformation (LFT) [14]. The size of the uncertainty required to cover a given set of models depends not only on the set of models, but also on the nominal model [15, 16]. In this paper, a method is proposed for finding the nominal model of an LFT uncertainty model that minimizes the ν-gap metric with respect to a set of known models. This is equivalent to finding a model that covers the set of known models with a minimum $\mathcal{H}_\infty$-norm uncertainty of the normalized coprime factors [17].

In the proposed optimization problem, every model in the model set gives rise to a bilinear matrix inequality (BMI), which the optimal solution has to satisfy. A difficulty is that BMI problems are not necessarily convex and they are much more difficult to handle computationally than linear matrix inequalities (LMIs) [18]. In this paper, the BMI problem is solved by a method developed in [19, 20] by linearizing the BMIs to obtain LMIs, which make the optimization problem convex. The iterative solution using the LMIs converges to the optimal solution satisfying the original BMIs. Because the solution is obtained as the frequency response at selected frequencies, the final model is determined by fitting a model to the frequency responses. A state-space model is used because the fitting can then easily be done subject to the same BMIs/LMIs to guarantee an optimal model.

The procedure is illustrated by an application to uncertainty modeling of the product composition dynamics of a distillation column.

II. Preliminaries

A. Uncertainty Models of LFT Type

An uncertainty model expressed as a linear fractional transformation has the general form

$$G_\Delta = G + H_{21}\Delta(I - H_{11}\Delta)^{-1}H_{12}, \qquad (1)$$

where G_Δ is a transfer function of the true system, G is the transfer function of a nominal model, and Δ is the transfer

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function of an unknown bounded perturbation causing uncertainty about the true system. Depending on the particular type of uncertainty (e.g., additive, input or output multiplicative, inverse types of uncertainty, combinations of various types of uncertainty), the matrices H_{11}, H_{12} and H_{21} are constant matrices or various combinations of constant matrices and the nominal transfer function. Well-posedness (stability) of the model requires $\|H_{11}\Delta\|_\infty < 1$. In general, the uncertainty Δ may be structured or unstructured, but in this paper only unstructured uncertainty is considered.

Since the ν-gap metric has a clear interpretation for a normalized coprime-factorized model, it is of interest to consider such a model. An uncertain normalized right-coprime factorized model has the form

$$G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}, \tag{2}$$

$$M^*M + N^*N = I, \tag{3}$$

where Δ_M and Δ_N are (unstructured) uncertainties, $G = NM^{-1}$ is the coprime-factorized nominal model, and (3) is the normalization of the coprime transfer functions M and N. The superscript *, which for a complex matrix denotes its complex conjugate transpose, has the meaning $G^*(s) = G^{\mathrm{T}}(-s)$ for a transfer function, where superscript T denotes the transpose of a matrix. Equation (2) is not in the form of (1), but by rearrangements it can be put into the form of (1) with

$$H_{11} = \begin{bmatrix} M^{-1} & 0 \end{bmatrix}, \quad H_{12} = M^{-1},$$
$$H_{21} = \begin{bmatrix} -G & I \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix}. \tag{4}$$

B. The ν-Gap Metric

The ν-gap metric, which was introduced as a metric on the set of finite-dimensional, linear time-invariant (LTI) systems, is a measure of the worst-case difference in performance of a feedback system expressed in terms of the corresponding open-loop systems [6, 7]. Thus, it is a useful tool for uncertainty and robustness studies of feedback systems.

The ν-gap metric can be defined in terms of ordinary open-loop transfer functions or the normalized coprime factors of these transfer functions. Selecting G and G_Δ as two transfer functions of interest, the ν-gap distance between them can be expressed as [7, 14]

$$\delta_\nu(G, G_\Delta) = \begin{cases} \|\Psi(G, G_\Delta)\|_\infty, & \text{if } \begin{cases} \det(I + G_\Delta^* G) \neq 0 \ \ \forall\omega \\ \mathrm{wno}\det(I + G_\Delta^* G) + \eta(G), \\ -\eta(G_\Delta) - \eta_0(G_\Delta) = 0 \end{cases} \\ 1, & \text{otherwise} \end{cases} \tag{5}$$

where

$$\Psi(G, G_\Delta) = (I + G_\Delta G_\Delta^*)^{-1/2}(G - G_\Delta)(I + G^*G)^{-1/2}. \tag{6}$$

Here η denotes the number of open right half-plane (RHP) poles, η_0 denotes the number of poles on the imaginary axis, wno denotes the "winding number", which is the number of counterclockwise encirclements of the origin evaluated on the Nyquist contour. The $\mathcal{H}_\infty$ norm $\|\Psi(G, G_\Delta)\|_\infty$ is defined

$$\|\Psi(G, G_\Delta)\|_\infty := \sup_\omega \bar{\sigma}\big(\Psi(G(\mathrm{j}\omega), G_\Delta(\mathrm{j}\omega))\big), \tag{7}$$

where $\bar{\sigma}(\Psi)$ is the maximum singular value of Ψ and ω denotes angular frequency ($s = \mathrm{j}\omega$).

As a demonstration of the usefulness of the ν-gap metric, consider the transfer functions [7]

$$G_\mathrm{A} = \frac{100}{2s+1}, \quad G_\mathrm{B} = \frac{100}{2s-1}, \quad G_\mathrm{C} = \frac{100}{(s+1)^2}. \tag{8}$$

The open-loop step responses of G_A and G_B are very different (G_B is unstable), whereas the step responses of G_A and G_C are quite similar. However, when the systems are controlled by a unity-feedback P controller, the setpoint responses of G_A and G_B are almost identical, whereas G_C oscillates heavily. The closed-loop result is implied by the ν-gap metric [14]: $\delta_\nu(G_\mathrm{A}, G_\mathrm{B}) = 0.020$, $\delta_\nu(G_\mathrm{A}, G_\mathrm{C}) = 0.899$, $\delta_\nu(G_\mathrm{B}, G_\mathrm{C}) = 0.894$.

III. Problem Formulation

Assume that a number of transfer function models G_k, $k = 1, \ldots, K$, have been determined for an uncertain system. Thus, $G_\Delta = \{G_1, \ldots, G_K\}$. It is desired to determine a nominal model G such that the worst-case distance between G and G_Δ is minimized in the ν-gap metric. If the models are expressed as normalized right-coprime factorizations, this is equivalent to minimizing $\|\Delta\|_\infty$ of the uncertainty defined in (4) [3, 17].

The worst-case distance in the ν-gap metric between a nominal model G and the model set G_Δ is

$$J_\nu(G) = \max_k \delta_\nu(G, G_k). \tag{9}$$

The nominal model yielding the smallest $J_\nu(G)$ is the best according to the ν-gap metric. If $\delta_\nu(G, G_k) = \|\Psi(G, G_k)\|_\infty$ is assumed to apply in all relevant cases, i.e., the if-part in (5) is assumed to apply, $J_\nu(G)$ is minimized by minimizing $\alpha > 0$ subject to $\max_k \|\Psi(G, G_k)\|_\infty \leq \sqrt{\alpha}$. This, in turn, is equivalent to minimization of α subject to

$$\max_{k,\omega} \Psi(G, G_k)\Psi(G, G_k)^* \preccurlyeq \alpha I, \tag{10}$$

where the notation $A \preccurlyeq B$ denotes that $A - B$ is a negative semidefinite matrix.

The solution to the optimization problem gives not only the nominal model G, but also the ν-gap measure

$$\delta_\nu(G, G_\Delta) = \sqrt{\alpha}. \tag{11}$$

It is not attempted to find $G(s)$ directly by minimization of α subject to (10). Instead, α is minimized frequency by frequency to yield optimal frequency responses $G_\Omega(\mathrm{j}\omega)$ of $G(s)$ at selected frequencies $\omega \in \Omega$. Here, $\mathrm{j} = \sqrt{-1}$ is the imaginary unit. After this, a model $G(s)$ is determined to fit the calculated frequency responses $G_\Omega(\mathrm{j}\omega)$ with a minimum α satisfying (10).

IV. Solution

A. Calculation Frequency by Frequency

Substitution of (6) into (10), with G replaced by G_Ω, gives an expression, which by the Schur-complement lemma [18] can be expressed as

$$\begin{bmatrix} \alpha I & (I + G_k G_k^*)^{-1/2}(G_\Omega - G_k) \\ (*) & I + G_\Omega^* G_\Omega \end{bmatrix} \succcurlyeq 0 \,, \forall k, \forall \omega \,, \qquad (12)$$

where (*) denotes the complex-conjugate transpose of the elements in the symmetrical position of the full matrix. When α and $G_\Omega(\mathrm{j}\omega)$ are decision variables, this is a BMI because of the term $G_\Omega^* G_\Omega$.

B. Linearization of the Bilinear Matrix Inequality

Assume that a suboptimal solution $\bar{G}_\Omega$ has been found. Linearization of $G_\Omega^* G_\Omega$ around this solution gives

$$\begin{aligned} G_\Omega^* G_\Omega &\approx \bar{G}_\Omega^* \bar{G}_\Omega + (G_\Omega^* - \bar{G}_\Omega^*)\bar{G}_\Omega + \bar{G}_\Omega^*(G_\Omega - \bar{G}_\Omega) \\ &= G_\Omega^* \bar{G}_\Omega + \bar{G}_\Omega^* G_\Omega - \bar{G}_\Omega^* \bar{G}_\Omega . \end{aligned} \qquad (13)$$

Substitution of (13) into (12) gives

$$\begin{bmatrix} \alpha I & (I + G_k G_k^*)^{-1/2}(G_\Omega - G_k) \\ (*) & I - \bar{G}_\Omega^* \bar{G}_\Omega + G_\Omega^* \bar{G}_\Omega + \bar{G}_\Omega^* G_\Omega \end{bmatrix} \succcurlyeq 0 \,, \forall k, \forall \omega \,. \qquad (14)$$

This is now an LMI when α and G_Ω are decision variables.

The second-order deviation neglected in the linearization is the matrix $(G_\Omega^* - \bar{G}_\Omega^*)(G_\Omega - \bar{G}_\Omega)$, which is positive (semi)definite. From this it follows that

$$G_\Omega^* G_\Omega \succcurlyeq G_\Omega^* \bar{G}_\Omega + \bar{G}_\Omega^* G_\Omega - \bar{G}_\Omega^* \bar{G}_\Omega . \qquad (15)$$

This means that (14) is a tighter constraint than the exact constraint (12) as long as the solution is suboptimal. This, in turn, means that the solution of an iteration using (14) will always satisfy (12). Thus, for every frequency there is a smaller α than the current solution $\alpha = \bar{\alpha}$ that will satisfy (12) tightly. Since (14) is equivalent to (12) when $G_\Omega = \bar{G}_\Omega$, a smaller α will be found when the iteration is continued with (14). From this it follows that the solution will converge to the optimal solution satisfying (12).

C. Finding a Nominal Model

It is desired to find a model $G(s)$ that closely fits the calculated frequency responses $G_\Omega(\mathrm{j}\omega)$, $\omega \in \Omega$. Since the fit is very unlikely to be perfect at every frequency, the constraint corresponding (12), i.e.,

$$\begin{bmatrix} \alpha I & (I + G_k G_k^*)^{-1/2}(G - G_k) \\ (*) & I + G^* G \end{bmatrix} \succcurlyeq 0 \,, \forall k, \forall \omega \,, \qquad (16)$$

will be violated with the already determined α, or satisfied only loosely, at a number of frequencies. It is straightforward to determine a new α at every considered frequency to tightly satisfy (16) with the model $G(s)$.

However, a better approach is to do a partial re-optimization by using parameters of $G(\mathrm{j}\omega)$ that appear linearly in the constraint corresponding to (14), i.e.,

$$\begin{bmatrix} \alpha I & (I + G_k G_k^*)^{-1/2}(G - G_k) \\ (*) & I - \bar{G}^* \bar{G} + G^* \bar{G} + \bar{G}^* G \end{bmatrix} \succcurlyeq 0 \,, \forall k, \forall \omega \,, \qquad (17)$$

as new decision variables. This will generally give smaller α values than the previous approach. Suitable parameters are the numerator parameters of the transfer functions of $G(s)$, e.g., the static gains. If $G(s)$ is parameterized as a state-space model,

$$G(s) = C(sI - A)^{-1} B + D \,, \qquad (18)$$

the parameters of D and either B or C can be used as decision variables.

There are many available methods for fitting $G(s)$ to $G_\Omega(\mathrm{j}\omega)$. If the model set $G_\Delta(s)$ has been determined through system identification, the used excitation signals may be known. If u_Δ represents the excitations for all experiments taken in sequence, it is well motivated to calculate the corresponding outputs as

$$y_\Omega(\mathrm{j}\omega) = G_\Omega(\mathrm{j}\omega) u_\Delta(\mathrm{j}\omega) \,, \ \forall \omega \,. \qquad (19)$$

A model can then be determined by frequency-domain system identification techniques using the input-output data $\{u_\Delta(\mathrm{j}\omega), y_\Omega(\mathrm{j}\omega), \forall \omega \in \Omega\}$. A re-optimization to ensure that (17) is satisfied (tightly) at every frequency can be done as described above.

V. Experimental Application

In this section, the presented procedure for determining an uncertainty model such that the nominal model minimizes the ν-gap metric is illustrated by an application to a pilot-scale distillation column.

A. Set of Models Determined by System Identification

A distillation column is a multivariable system usually characterized by a strong directionality, which means that the transfer matrix is ill-conditioned and nearly singular. This makes the system difficult to control as well as to identify. In system identification it is therefore important to excite all directions sufficiently, especially the low-gain direction. Because it may be difficult to capture all relevant dynamics of a nonlinear process in a single linear model, an appealing approach is to determine a set of (linear) models [21].

In this application, the set of linear models determined in [21] for "operating point B" is used to calculate the best nominal model according to the ν-gap metric. To determine the models, the distillation column was excited by a series of consecutive step changes in the high- and low-gain directions. Even though the information contained in a step response is quite limited, it was possible to determine a second-order plus time-delay model for each input-output pair in a 2×2 model by considering known properties of high- and low-gain distillation data. Six such models, $G_k(s)$, $k=1,\ldots,6$, were determined. A nominal model $G_0(s)$ was also determined by a well-motivated combination of the individual models. It is of interest to compare this nominal model with the one to be determined in this paper.

In [21], the transfer functions are expressed as the sum of two first-order transfer functions, one pertaining to the high-gain direction, the other to the low-gain direction. For convenience, the second-order transfer functions are here expressed in a standard way, i.e.,

$$g(s)=\frac{\kappa(\tau_3 s+1)\mathrm{e}^{-\theta s}}{(\tau_1 s+1)(\tau_2 s+1)}. \tag{20}$$

The parameter values for the nominal model and the six individual models are given in Table 1. Duplicated time constants are due to certain directionality assumptions and the fact that the models G_1, G_2 and G_3 were determined from excitations mainly in the low-gain direction, the models G_4, G_5 and G_6 from excitations mainly in the high-gain direction. The seemingly high accuracy of the gains follows from addition of a high and a low gain. The time unit is minutes.

To illustrate the differences between the identified models, magnitude plots of the four transfer functions for every model are shown in Fig. 1 (using a lin-log plot to better differentiate between the graphs).

TABLE I. IDENTIFIED MODEL SET

$G_k(s)$	$g_{ij}(s)$	κ	τ_1	τ_2	τ_3	θ
$G_0(s)$	$g_{11}(s)$	–0.04229	25.16	5.55	8.10	1.5
	$g_{12}(s)$	0.09349	25.16	5.55	5.11	1.5
	$g_{21}(s)$	0.11733	30.98	3.60	4.66	3.5
	$g_{22}(s)$	–0.27858	30.98	3.60	3.42	3.0
$G_1(s)$	$g_{11}(s)$	–0.04084	25.16	5.23	7.20	1.5
	$g_{12}(s)$	0.09404	25.16	5.23	4.90	1.5
	$g_{21}(s)$	0.11620	30.98	1.35	2.22	3.5
	$g_{22}(s)$	–0.27903	30.98	1.35	1.20	3.0
$G_2(s)$	$g_{11}(s)$	–0.04213	25.16	5.49	7.98	1.5
	$g_{12}(s)$	0.09354	25.16	5.49	5.06	1.5
	$g_{21}(s)$	0.11786	30.98	6.23	7.29	3.5
	$g_{22}(s)$	–0.27836	30.98	6.23	6.05	3.0
$G_3(s)$	$g_{11}(s)$	–0.04256	25.16	7.79	10.14	1.5
	$g_{12}(s)$	0.09339	25.16	7.79	7.38	1.5
	$g_{21}(s)$	0.11847	30.98	1.38	2.80	3.5
	$g_{22}(s)$	–0.27812	30.98	1.38	1.14	3.0
$G_4(s)$	$g_{11}(s)$	–0.04249	33.16	5.55	9.12	1.5
	$g_{12}(s)$	0.09389	33.16	5.55	4.93	1.5
	$g_{21}(s)$	0.17253	45.61	3.60	4.70	5.5
	$g_{22}(s)$	–0.41578	45.61	3.60	3.42	5.0
$G_5(s)$	$g_{11}(s)$	–0.04239	25.32	5.55	8.11	4.0
	$g_{12}(s)$	0.09379	25.32	5.55	5.11	4.0
	$g_{21}(s)$	0.13313	30.59	3.60	4.52	3.5
	$g_{22}(s)$	–0.31778	30.59	3.60	3.45	3.0
$G_6(s)$	$g_{11}(s)$	–0.06149	47.82	5.55	9.32	6.0
	$g_{12}(s)$	0.14319	47.82	5.55	4.93	6.0
	$g_{21}(s)$	0.11573	38.11	3.60	4.95	3.5
	$g_{22}(s)$	–0.27468	38.11	3.60	3.37	3.0

B. Minimization of the ν-Gap Metric

Optimal frequency responses, $G_\Omega(\mathrm{j}\omega)$, are determined by iteratively minimizing α subject to (14) for $\forall\omega\in\Omega$. The optimization is done with MATLAB and the YALMIP software [22]. The selected frequency points are 99 logarithmically equal-spaced points in the range $[0.001, 2\pi]$ rad/min, 2π rad/min being the Nyquist frequency because the sampling interval is 0.5 min. The iterations are started with $\bar{G}_\Omega(\mathrm{j}\omega)=G_0(\mathrm{j}\omega)$.

Magnitude plots of the obtained optimal frequency responses are shown in Fig. 1 for $\forall\omega\in\Omega$. Plots of $\sqrt{\alpha}$ as function of frequency are shown in Fig. 2 for G_0 and G_Ω, the maximum value of $\sqrt{\alpha}$ being the ν-gap measure. Naturally, G_Ω has a smaller ν-gap measure than G_0.

C. Fitting a State-Space Model

The model set was identified from data obtained by a sequence of step changes, $u_1,\ldots,u_6$, as described in [21]. Because the duration of each step was 30 min, the frequency response of the whole input sequence is

$$\begin{aligned} u_\Lambda(\mathrm{j}\omega)=u_1(\mathrm{j}\omega)+u_2(\mathrm{j}\omega)\mathrm{e}^{-30\omega\mathrm{j}}+\cdots \\ u_3(\mathrm{j}\omega)\mathrm{e}^{-60\omega\mathrm{j}}+u_4(\mathrm{j}\omega)\mathrm{e}^{-90\omega\mathrm{j}}+\cdots \\ u_5(\mathrm{j}\omega)\mathrm{e}^{-120\omega\mathrm{j}}+u_6(\mathrm{j}\omega)\mathrm{e}^{-150\omega\mathrm{j}}. \end{aligned} \tag{21}$$

The corresponding output $y_\Omega(\mathrm{j}\omega)$ is calculated according to (19).

A state-space model is identified from the input-output data $\{u_\Lambda(\mathrm{j}\omega), y_\Omega(\mathrm{j}\omega), \forall\omega\in\Omega\}$ using the MATLAB *Control System Toolbox* [23] and the subspace algorithm N4SID with default toolbox options. It turns out that it is difficult to obtain a good fit to the data with any model order. Thus, there is no incentive to choose a high-order model and a 4th order model is therefore selected. The obtained model is

$$\begin{aligned} \dot{x}(t)&=Ax(t)+Bu(t) \\ y(t)&=Cx(t)+Du(t) \end{aligned} \tag{22}$$

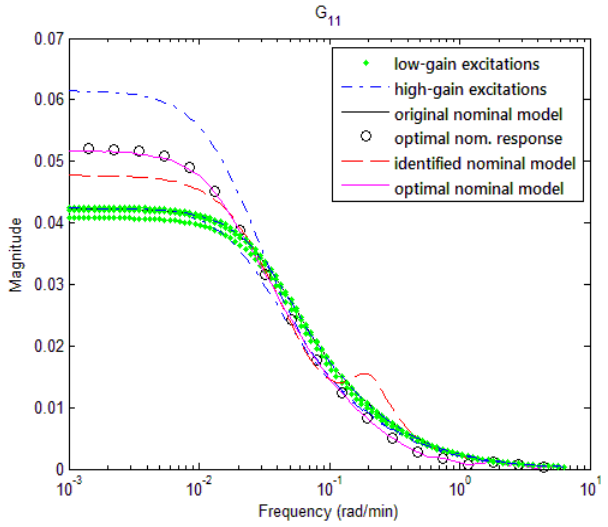

Figure 1a. Magnitude plots of G_{11}.

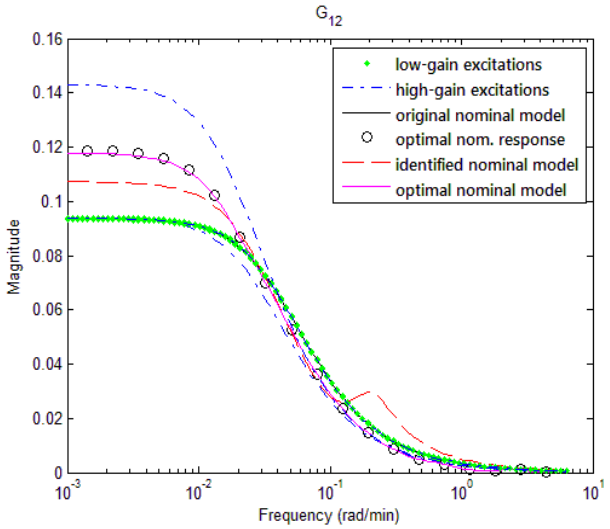

Figure 1b. Magnitude plots of G_{12}.

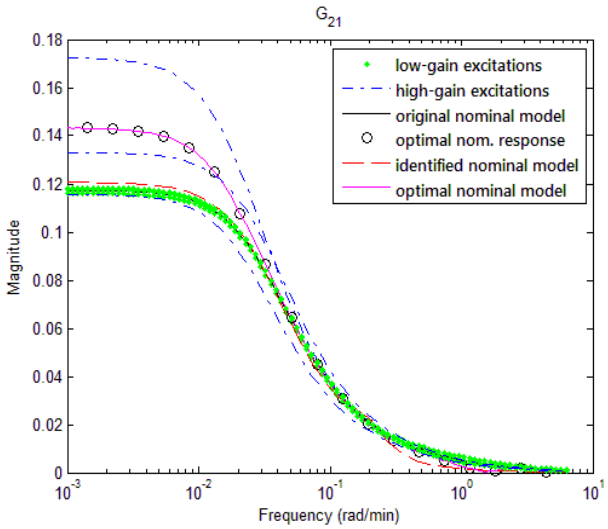

Figure 1c. Magnitude plots of G_{21}.

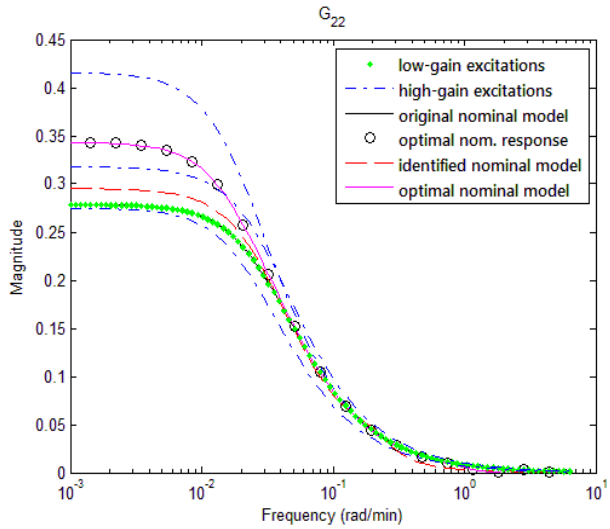

Figure 1d. Magnitude plots of G_{22}.

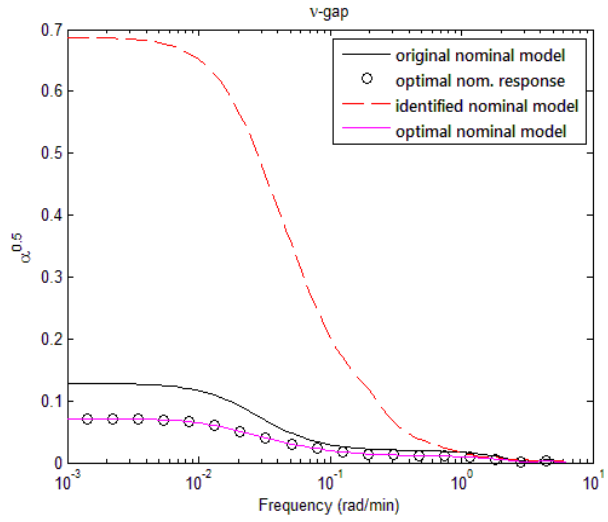

Figure 2. Plots of ν-gap vs frequency.

with

$$A=\begin{bmatrix} -0.0165 & 0.3724 & -1.0932 & 0.1491 \\ -0.0029 & -0.1509 & 0.1811 & 0.9366 \\ 0.0010 & -0.0074 & -0.1651 & 0.7501 \\ 0.0000 & 0.0027 & -0.0600 & 0.0064 \end{bmatrix},\quad B=\begin{bmatrix} 0.0078 & -0.0174 \\ -0.0058 & 0.0140 \\ 0.0001 & -0.0001 \\ 0.0008 & -0.0015 \end{bmatrix}$$

$$C=\begin{bmatrix} -0.1306 & 0.2382 & 0.3302 & 0.5765 \\ 0.3774 & 0.2741 & -0.4319 & 0.1816 \end{bmatrix},\quad D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (23)$$

The magnitude plots in Fig.1 show that the model is not very good. The plot of $\sqrt{\alpha}$ shown in Fig. 2 is also much higher than the plot for the calculated $G_{\Omega}(\mathrm{j}\omega)$.

However, this is not the final model. It can be improved dramatically by minimizing α with B or C as a decision variable keeping the other state-space parameters intact. The minimization is done subject to (17) with G replaced by (18). Optimization on B yields the solution

$$B = \begin{bmatrix} -0.0196 & -0.0168 \\ -0.0039 & -0.0063 \\ -0.0058 & -0.0002 \\ 0.0008 & -0.0008 \end{bmatrix}. \quad (24)$$

As can be seen from Fig. 2, where $\sqrt{\alpha}$ is plotted for this solution, it exactly matches the optimal solution. Magnitude plots of the corresponding transfer function are shown in Fig. 1.

VI. Conclusion

A method was presented for determining frequency responses of a model such that the distance between the model and a set of known models is minimized in the ν-gap metric. The optimization problem involves bilinear matrix inequalities (BMIs), which make the problem non-convex. The BMIs were linearized to enable an iterative solution, where each optimization step is convex. It was shown that the linear matrix inequalities (LMIs) resulting from the linearization are more restrictive than the corresponding BMIs. The solution in every iterative step is thus an upper bound on the optimal solution. It was shown that the solution converges to the optimal one restricted by the BMIs

At this point, the solution was in the form of frequency responses calculated frequency by frequency. A parametric solution was obtained by determining a state-space model that matched the optimal solution at the frequency points used in the previous step. This solution minimizes the ν-gap distance between the model and the set of known models. From a feedback perspective, the model is an optimal nominal model in a normalized coprime-factorized uncertainty model that tightly covers the set of known models. The model is thus suitable for robust control design.

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