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Vostrikova, Liudmila; Salminen, Paavo

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On moments of integral exponential functionals of additive processes

Paavo Salminen
Åbo Akademi University,
Faculty of Science and Engineering,
FIN-20500 Åbo, Finland,
phsalmin@abo.fi

Lioudmila Vostrikova
Université d'Angers,
Département de Mathématiques,
F-49045 Angers Cedex 01, France,
vostrik@univ-angers.fr

Abstract

For real-valued additive process $(X_t)_{t \geq 0}$ a recursive equation is derived for the entire positive moments of functionals

$$I_{s,t} = \int_s^t \exp(-X_u) du, \quad 0 \leq s < t \leq \infty,$$

in case the Laplace exponent of X_t exists for positive values of the parameter. From the equation emerges an easy-to-apply sufficient condition for the finiteness of the moments. As an application we study first hit processes of diffusions.

Keywords: independent increments, Lévy process, subordinator, Bessel process, geometric Brownian motion

AMS Classification: 60J75, 60J60. 60E10

1 Introduction

Let $X = (X_t)_{t \geq 0}$, $X_0 = 0$, be a real valued additive process, i.e., a strong Markov process with independent increments having càdlàg sample paths which are continuous in probability (cf. Sato [12], p.3). Important examples of additive processes are:

- (a) Deterministic time transformations of Lévy processes, that is, if $(L_s)_{s \geq 0}$ is a Lévy process and $s \mapsto g(s)$ is an increasing continuous function such that $g(0) = 0$ then $(L_{g(s)})_{s \geq 0}$ is an additive process.
- (b) Integrals of deterministic functions with respect to a Lévy process, that is, if $(L_s)_{s \geq 0}$ is a Lévy process and $s \mapsto g(s)$ is a measurable and locally bounded function then

$$Z_t := \int_0^t g(s) dL_s, \quad t \geq 0,$$

is an additive process.

- (c) First hit processes of one-dimensional diffusions, that is, if $(Y_s)_{s \geq 0}$ is a diffusion taking values in $[0, \infty)$, starting from 0, and drifting to $+\infty$ then

$$H_a := \inf\{t : Y_t > a\}, \quad a \geq 0,$$

is an additive process.

Of course, Lévy processes themselves constitute a large and important class of additive processes.

The aim of this paper is to study integral exponential functionals of X , i.e., functionals of the form

$$I_{s,t} := \int_s^t \exp(-X_u) du, \quad 0 \leq s < t \leq \infty, \quad (1)$$

in particular, the moments of $I_{s,t}$. We refer also to a companion paper [10], where stochastic calculus is used to study the Mellin transforms of $I_{s,t}$ when the underlying additive process is a semimartingale with absolutely continuous characteristics.

The main result of the paper is a recursive equation, see (5) below, which generalizes the formula for Lévy processes presented in Urbanik [13] and Carmona, Petit and Yor [6], see also Bertoin and Yor [3]. This formula for Lévy processes is also displayed below in (18). In Epifani, Lijoi and

Prünster [8] an extension of the Lévy process formula to integral functionals up to $t = \infty$ of increasing additive processes is discussed, and their formula (7) can be seen as a special case of our formula (15) – as we found out after finishing our work. We refer to these papers and also to [10] for further references and applications, e.g., in financial mathematics and statistics.

In spite of the existing closely related results we feel that it is worthwhile to provide a more thorough discussion of the topic. We also give new (to our best knowledge) applications of the formulas for first hit processes of diffusions and present explicit results for Bessel processes and geometric Brownian motions.

2 Main results

Let $(X_t)_{t \geq 0}$ be an additive process and define for $0 \leq s \leq t \leq \infty$ and $\alpha \geq 0$

$$m_{s,t}^{(\alpha)} := \mathbf{E} (I_{s,t}^\alpha) = \mathbf{E} \left(\left(\int_s^t e^{-X_u} du \right)^\alpha \right), \quad \alpha \geq 0, \quad (2)$$

and

$$m_t^{(\alpha)} := m_{0,t}^{(\alpha)}, \quad m_\infty^{(\alpha)} := m_{0,\infty}^{(\alpha)}.$$

In this section we derive a recursive integral equation for $m_{s,t}^{(\alpha)}$ under the following assumption:

$$(A) \quad \mathbf{E}(e^{-\lambda X_t}) < \infty \text{ for all } t \geq 0 \text{ and } \lambda \geq 0.$$

Under this assumption we define

$$\Phi(t; \lambda) := -\log \mathbf{E}(e^{-\lambda X_t}). \quad (3)$$

Since X is assumed to be continuous in probability it follows that $t \mapsto \Phi(t, \lambda)$ is continuous. Moreover, $X_0 = 0$ a.s. implies that $\Phi(0; \lambda) = 0$. If X is a Lévy process satisfying (A) we write (with a slight abuse of the notation)

$$\mathbf{E}(e^{-\lambda X_t}) = e^{-t\Phi(\lambda)}. \quad (4)$$

See Sato [12] Theorem 9.8, p.52, for properties and the structure of the Laplace exponent Φ of the infinitely divisible distribution X_t .

In particular, (A) is valid for increasing additive processes. Important examples of these are the first hit processes for diffusions (cf. (c) in Introduction). Assumption (A) holds also for additive processes of type (a) in Introduction if the underlying Lévy process fullfills (A).

Remark 2.1. *If X is a semimartingale with absolutely continuous characteristics, a sufficient condition for the existence of the Laplace exponent as in (A) in terms of the jump measure is given in Proposition 1 in [10].*

By continuity we have – from Jensen’s inequality – the following result

Lemma 2.2. *Under assumption (A) $m_{s,t}^{(\alpha)} < \infty$ for all $0 \leq \alpha \leq 1$ and $0 \leq s \leq t < \infty$.*

The main result of the paper is given in the next theorem. In the proof we are using similar ideas as in [6].

Theorem 2.3. *Under assumption (A) it holds for $\alpha \geq 1$ and $0 \leq s \leq t < \infty$ that the moments $m_{s,t}^{(\alpha)}$ are finite and satisfy the recursive equation*

$$m_{s,t}^{(\alpha)} = \alpha \int_s^t m_{u,t}^{(\alpha-1)} e^{-(\Phi(u;\alpha) - \Phi(u;\alpha-1))} du. \quad (5)$$

Proof. We start with by noting that the function $s : [0, t] \mapsto I_{s,t}$ is for any $t > 0$ continuous and strictly decreasing. Hence, for $\alpha \geq 1$

$$\alpha \int_0^s I_{u,t}^{\alpha-1} dI_{u,t} = \alpha \int_{I_{0,t}}^{I_{s,t}} v^{\alpha-1} dv = I_{s,t}^\alpha - I_{0,t}^\alpha,$$

where the integral is a pathwise (a.s.) Riemann-Stieltjes integral and the formula for the change of variables (see, e.g., Apostol [4], p. 144) is used. Consequently, from the definition of $I_{s,t}$ it follows that

$$I_{s,t}^\alpha - I_{0,t}^\alpha = -\alpha \int_0^s I_{u,t}^{\alpha-1} e^{-X_u} du = -\alpha \int_0^s I_{u,t}^{\alpha-1} e^{-X_u} du.$$

Introducing the shifted functional $\widehat{I}_{s,t}$ via

$$\widehat{I}_{s,t} := \int_0^{t-s} e^{-(X_{u+s} - X_s)} du$$

we have

$$\widehat{I}_{s,t} = e^{X_s} I_{s,t} = e^{X_s} \int_s^t e^{-X_u} du, \quad (6)$$

and, therefore,

$$I_{s,t}^\alpha - I_{0,t}^\alpha = -\alpha \int_0^s \widehat{I}_{u,t}^{\alpha-1} e^{-\alpha X_u} du. \quad (7)$$

Notice that the independence of increments implies that $\widehat{I}_{u,t}^{\alpha-1}$ and $e^{-\alpha X_u}$ are independent, and, hence, for all $\alpha \geq 1$

$$\mathbf{E} \left(\widehat{I}_{u,t}^\alpha \right) = \mathbf{E} \left(I_{u,t}^\alpha \right) / \mathbf{E} \left(e^{-\alpha X_u} \right) \quad (8)$$

Then evoking Lemma 2.2 and (8) yield for $0 \leq \alpha \leq 1$ and $0 \leq s \leq t < \infty$

$$\mathbf{E} \left(\widehat{I}_{s,t}^\alpha \right) \leq \mathbf{E} \left(I_{0,t}^\alpha \right) / \mathbf{E} \left(e^{-\alpha X_u} \right) < \infty. \quad (9)$$

Assume now that $\alpha \in [1, 2]$. Taking the expectations in (7) and applying Fubini's theorem gives

$$\mathbf{E} \left(I_{s,t}^\alpha - I_{0,t}^\alpha \right) = -\alpha \int_0^s \mathbf{E} \left(\widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} \left(e^{-\alpha X_u} \right) du > -\infty \quad (10)$$

where the finiteness follows from (9). Since $I_{s,t} \rightarrow 0$ a.s. when $s \uparrow t$ we obtain by applying monotone convergence in (10)

$$\mathbf{E} \left(I_{0,t}^\alpha \right) = \alpha \int_0^t \mathbf{E} \left(\widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} \left(e^{-\alpha X_u} \right) du < \infty. \quad (11)$$

Putting (10) and (11) together results to the equation

$$\mathbf{E} \left(I_{s,t}^\alpha \right) = \alpha \int_s^t \mathbf{E} \left(\widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} \left(e^{-\alpha X_u} \right) du. \quad (12)$$

Finally, using (8) and (9) in (12) and recalling (3) yield (5) for $\alpha \in [1, 2]$. Since (8) is valid for all α and, as just proved, the finiteness holds for $\alpha \in [1, 2]$ the proof of (5) for arbitrary $\alpha > 2$ is easily accomplished by induction. \square

Corollary 2.4. *Let $(X_t)_{t \geq 0}$ be a Lévy process with the Laplace exponent as in (4). Then (5) with $s = 0$ and $t < \infty$ is equivalent to*

$$m_t^{(\alpha)} = \alpha e^{-t\Phi(\alpha)} \int_0^t m_u^{(\alpha-1)} e^{u\Phi(\alpha)} du. \quad (13)$$

Proof. Put $s = 0$ in (5) to obtain

$$m_t^{(\alpha)} = \alpha \int_0^t m_{u,t}^{(\alpha-1)} e^{-u(\Phi(\alpha) - \Phi(\alpha-1))} du. \quad (14)$$

Consider

$$\begin{aligned}
m_{u,t}^{(\alpha-1)} &= \mathbf{E} \left(\left(\int_u^t e^{-X_v} dv \right)^{\alpha-1} \right) \\
&= \mathbf{E} \left(e^{-(\alpha-1)X_u} \left(\int_u^t e^{-(X_v-X_u)} dv \right)^{\alpha-1} \right) \\
&= e^{-u\Phi(\alpha-1)} \mathbf{E} \left(\left(\int_0^{t-u} e^{-(X_v+u-X_u)} dv \right)^{\alpha-1} \right) \\
&= e^{-u\Phi(\alpha-1)} \mathbf{E} \left(\left(\int_0^{t-u} e^{-X_v} dv \right)^{\alpha-1} \right) \\
&= e^{-u\Phi(\alpha-1)} m_{t-u}^{(\alpha-1)}.
\end{aligned}$$

Substituting this expression into (14) proves the claim. \square

For positive integer values on α the recursive equation (5) can be solved explicitly to obtain the formula (15) in the next proposition. However, we offer another proof highlighting the symmetry properties present in the expressions of the moments of the exponential functional.

Proposition 2.5. *For $0 \leq s \leq t \leq \infty$ and $n = 1, 2, \dots$ it holds*

$$\begin{aligned}
m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \\
&\quad \cdots \int_{t_{n-1}}^t dt_n \exp \left(- \sum_{k=1}^n (\Phi(t_k; n-k+1) - \Phi(t_k; n-k)) \right).
\end{aligned} \tag{15}$$

In particular, $m_{s,\infty}^{(n)} < \infty$ if and only if the multiple integral on the right hand side of (15) is finite.

Proof. Let $t < \infty$ and consider

$$\begin{aligned}
m_{s,t}^{(n)} &= \mathbf{E} \left(\left(\int_s^t e^{-X_u} du \right)^n \right) \\
&= \mathbf{E} \left(\int_s^t \cdots \int_s^t e^{-X_{t_1} - \cdots - X_{t_n}} dt_1 \cdots dt_n \right) \\
&= n! \mathbf{E} \left(\int_s^t dt_1 e^{-X_{t_1}} \int_{t_1}^t dt_2 e^{-X_{t_2}} \cdots \int_{t_{n-1}}^t dt_n e^{-X_{t_n}} \right) \\
&= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \mathbf{E} \left(e^{-(X_{t_1} + \cdots + X_{t_n})} \right),
\end{aligned}$$

where, in the third step, we use that $(t_1, t_2, \dots, t_n) \mapsto e^{-(X_{t_1} + \dots + X_{t_n})}$ is symmetric. By the independence of the increments

$$\mathbf{E}(e^{-\alpha X_t}) = \mathbf{E}\left(e^{-\alpha(X_t - X_s) - \alpha X_s}\right) = \mathbf{E}\left(e^{-\alpha(X_t - X_s)}\right) \mathbf{E}\left(e^{-\alpha X_s}\right).$$

Consequently,

$$\mathbf{E}\left(e^{-\alpha(X_t - X_s)}\right) = \mathbf{E}\left(e^{-\alpha X_t}\right) / \mathbf{E}\left(e^{-\alpha X_s}\right) = e^{-(\Phi(t; \alpha) - \Phi(s; \alpha))}.$$

Since,

$$X_{t_1} + \dots + X_{t_n} = \sum_{k=1}^n (n - k + 1) (X_{t_k} - X_{t_{k-1}}), \quad t_0 := 0,$$

we have

$$\begin{aligned} m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \dots \\ &\dots \int_{t_{n-1}}^t dt_n \exp\left(-\sum_{k=1}^n (\Phi(t_k; n - k + 1) - \Phi(t_{k-1}; n - k + 1))\right). \end{aligned}$$

Applying the initial values given in (4) yields the claimed formula (15). The statement concerning the finiteness of $m_{s,\infty}^{(n)}$ follows by applying the monotone convergence theorem as $t \rightarrow \infty$ on both sides of (15). \square

Corollary 2.6. *Variable I_∞ has all the positive moments if for all n*

$$\int_0^\infty e^{-(\Phi(s;n) - \Phi(s;n-1))} ds < \infty. \quad (16)$$

Proof. From (15) we have

$$m_t^{(n)} \leq n! \prod_{k=1}^n \int_0^\infty e^{-(\Phi(s;n) - \Phi(s;n-1))} ds. \quad (17)$$

The right hand side of (17) is finite if (16) holds. Let $t \rightarrow \infty$ in (17). By monotone convergence, $m_\infty^{(n)} = \lim_{t \rightarrow \infty} m_t^{(n)}$, and the claim is proved. \square

Formula (18) below extends the corresponding formula for subordinators found [13], see also [3], p.195, for Lévy processes satisfying (A). It is a straightforward implication of Proposition 2.5.

Corollary 2.7. *Let $(X_t)_{t \geq 0}$ be a Lévy process with the Laplace exponent as in (4) and define $n^* := \min\{n \in \{1, 2, \dots\} : \Phi(n) \leq 0\}$. Then*

$$m_\infty^{(n)} := \mathbf{E}(I_\infty^n) = \begin{cases} \frac{n!}{\prod_{k=1}^n \Phi(k)}, & \text{if } n < n^*, \\ +\infty, & \text{if } n \geq n^*. \end{cases} \quad (18)$$

Example 2.8. A much studied functional is obtained when taking $X = (X_t)_{t \geq 0}$ with $X_t = \sigma W_t + \mu t$, $\sigma > 0, \mu > 0$, where $(W_t)_{t \geq 0}$ is a standard Brownian motion. In Dufresne [7] and Yor [14] (see also Salminen and Yor [11]) it is proved that

$$I_\infty := \int_0^\infty \exp(-(\sigma W_s + \mu s)) ds \sim H_0^{(\delta)}, \quad (19)$$

where H_0^δ is the first hitting time of 0 for a Bessel process of dimension $\delta = 2(1 - (\mu/\sigma^2))$ starting from $\sigma/2$, and \sim means "is identical in law with". In particular, it holds

$$\int_0^\infty \exp(-2W_s + \mu s) ds \sim \frac{1}{2Z_\mu}, \quad (20)$$

where Z_μ is a gamma-distributed random variable with rate 1 and shape $\mu/2$. We refer to [7] for a discussion showing how the functional on the left hand side of (19) arises as the present value of a perpetuity in a discrete model after a limiting procedure. Since the Lévy exponent in this case is

$$\Phi(\lambda) = \lambda\mu - \frac{1}{2}\lambda^2\sigma^2,$$

the criterium in Corollary 2.7 yields

$$\mathbf{E}(I_\infty^n) < \infty \Leftrightarrow n < 2\mu/\sigma^2,$$

which readily can also be checked from (20).

3 First hit processes of one-dimensional diffusions

We recall first some facts concerning the first hitting times of one-dimensional (or linear) diffusions. Let now $Y = (Y_s)_{s \geq 0}$ be a linear diffusion taking values in an interval I . To fix ideas assume that I equals \mathbf{R} or $(0, \infty)$ or $[0, \infty)$ and that

$$\limsup_{s \rightarrow \infty} Y_s = +\infty \quad \text{a.s.} \quad (21)$$

Assume $Y_0 = v$ and consider for $a \geq v$ the first hitting time

$$H_a := \inf\{s : Y_s > a\}.$$

Defining $X_t := H_{t+v}$, $t \geq 0$, it is easily seen – since Y is a strong Markov process – that $X = (X_t)_{t \geq 0}$ is an increasing purely discontinuous additive process starting from 0. Moreover, from (21) it follows that $X_t < \infty$ a.s. for all t . The process X satisfies (A) and it holds

$$\mathbf{E}_v(e^{-\beta X_t}) = \mathbf{E}_v(e^{-\beta H_{t+v}}) = \frac{\psi_\beta(v)}{\psi_\beta(t+v)}, \quad t \geq 0, \quad (22)$$

where $\beta \geq 0$, \mathbf{E}_v is the expectation associated with Y , $Y_0 = v$, and ψ_β is a unique (up to a multiple) positive and increasing solution (satisfying appropriate boundary conditions) of the ODE

$$(Gf)(x) = \beta f(x), \quad (23)$$

where G denotes the differential operator associated with Y . For details about diffusions (and further references), see Itô and McKean [9], and [5]. The Laplace transform of X_t can also be represented as follows

$$\mathbf{E}_v(e^{-\beta X_t}) = \exp\left(-\int_v^{t+v} S(du) \int_0^\infty (1 - e^{-\beta x})n(u, dx)\right), \quad (24)$$

where S is the scale function, and n is a kernel such that for all $v \in I$ and $t \geq 0$

$$\int_v^{t+v} \int_0^\infty (1 \wedge x)n(u, dx)S(du) < \infty.$$

Representation (24) clearly reveals the structure of X as a process with independent increments. From (22) and (24) we may conclude that

$$\int_0^\infty (1 - e^{-\beta x})n(u, dx) = \lim_{w \rightarrow u-} \frac{1 - \mathbf{E}_w(e^{-\beta X_u})}{S(u) - S(w)}. \quad (25)$$

We now pass to present examples of exponential functionals of first hit processes. Firstly, we study Bessel processes satisfying (21) and show, in particular, that the exponential functional of the first hit process has all the moments. In our second example it is seen that the exponential functional of the first hit process of geometric Brownian motion has only finitely many moments depending on the values of the parameters.

Example 3.1. Bessel processes. Let Y be a Bessel process starting from $v > 0$. The differential operator associated with Y is given by

$$(Gf)(x) = \frac{1}{2}f''(x) + \frac{\delta - 1}{2x}f'(x), \quad x > 0,$$

where $\delta \in \mathbf{R}$ is called the dimension parameter. From [5] we extract the following information

- for $\delta \geq 2$ the boundary point 0 is entrance-not-exit and (21) holds,
- for $0 < \delta < 2$ the boundary point 0 is non-singular and (21) holds when the boundary condition at 0 is reflection,
- for $\delta \leq 0$ (21) does not hold.

In case when (21) is valid the Laplace exponent for the first hit process $X = (X_t)_{t \geq 0}$ is given for $v > 0$ and $t \geq 0$ by

$$\mathbf{E}_v(e^{-\beta X_t}) = \frac{\psi_\beta(v)}{\psi_\beta(t)} = \frac{v^{1-\delta/2} I_{\delta/2-1}(v\sqrt{2\beta})}{t^{1-\delta/2} I_{\delta/2-1}((t+v)\sqrt{2\beta})}, \quad (26)$$

where \mathbf{E}_v is the expectation associated with Y when started from v and I denotes the modified Bessel function of the first kind. For simplicity, we wish to study the exponential functional of X when $v = 0$. To find the Laplace exponent when $v = 0$ we let $v \rightarrow 0$ in (26). For this, recall that for $p \neq -1, -2, \dots$

$$I_p(v) \simeq \frac{1}{\Gamma(p+1)} \left(\frac{v}{2}\right)^p \quad \text{as } v \rightarrow 0. \quad (27)$$

Consequently,

$$\begin{aligned} \mathbf{E}_0(e^{-\beta X_t}) &= \lim_{v \rightarrow 0} \mathbf{E}_v(e^{-\beta X_t}) \\ &= \frac{1}{\Gamma(\nu+1)} \left(\frac{\sqrt{2\beta}}{2}\right)^{\delta/2-1} \frac{t^{\delta/2-1}}{I_{\delta/2-1}(t\sqrt{2\beta})} \\ &=: e^{-\Phi(t;\beta)}. \end{aligned}$$

The validity of (16), that is, the finiteness of the positive moments, can now be checked by exploiting the asymptotic behaviour of I_p saying that for all $p \in \mathbf{R}$ (see Abramowitz and Stegun [1], 9.7.1 p.377)

$$I_p(t) \simeq e^t / \sqrt{2\pi t} \quad \text{as } t \rightarrow \infty. \quad (28)$$

Indeed, for $n = 1, 2, \dots$

$$\begin{aligned} e^{-(\Phi(t;n)-\Phi(t;n-1))} &= \frac{n^{\delta/2-1}}{I_{\delta/2-1}(t\sqrt{2n})} \frac{I_{\delta/2-1}(t\sqrt{2(n-1)})}{(n-1)^{\delta/2-1}} \\ &\simeq \left(\frac{n}{n-1}\right)^{\delta/2-1} \left(\frac{n}{n-1}\right)^{1/4} e^{-t(\sqrt{2n}-\sqrt{2(n-1)})}, \end{aligned}$$

which clearly is integrable at $+\infty$. Consequently, by Corollary 2.6, the integral functional

$$\int_0^\infty e^{-Xt} dt$$

has all the (positive) moments.

Example 3.2. Geometric Brownian motion. Let $Y = (Y_s)_{s \geq 0}$ be a geometric Brownian motion with parameters $\sigma^2 > 0$ and $\mu \in \mathbf{R}$, i.e.,

$$Y_s = \exp\left(\sigma W_s + \left(\mu - \frac{1}{2}\sigma^2\right)s\right)$$

where $W = (W_s)_{s \geq 0}$ is a standard Brownian motion initiated at 0. Since $W_s/s \rightarrow 0$ a.s. when $s \rightarrow \infty$ it follows

- $\lim_{s \rightarrow \infty} Y_s = +\infty$ a.s if $\mu > \frac{1}{2}\sigma^2$,
- $\lim_{s \rightarrow \infty} Y_s = 0$ a.s if $\mu < \frac{1}{2}\sigma^2$.
- $\limsup_{s \rightarrow \infty} Y_s = +\infty$ and $\liminf_{s \rightarrow \infty} Y_s = 0$ a.s. if $\mu = \frac{1}{2}\sigma^2$.

Consequently, condition (21) is valid if and only if $\mu \geq \frac{1}{2}\sigma^2$. Since $Y_0 = 1$ we consider the first hitting times of points $a \geq 1$. Consider

$$\begin{aligned} H_a &:= \inf\{s : Y_s = a\} \\ &= \inf\left\{s : \exp\left(\sigma W_s + \left(\mu - \frac{1}{2}\sigma^2\right)s\right) = a\right\} \\ &= \inf\left\{s : W_s + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} s = \frac{1}{\sigma} \log a\right\}. \end{aligned}$$

We assume now that $\sigma > 0$ and $\mu \geq \frac{1}{2}\sigma^2$. Let $\nu := (\mu - \frac{1}{2}\sigma^2)/\sigma$. Then H_a is identical in law with the first hitting time of $\log a/\sigma$ for Brownian motion with drift $\nu \geq 0$ starting from 0. Consequently, letting $X_t := H_{1+t}$ we have

for $t \geq 0$

$$\begin{aligned}
\mathbf{E}_1(e^{-\beta X_t}) &= \frac{\psi_\beta(0)}{\psi_\beta(\log(1+t)/\sigma)} \\
&= \exp\left(-\left(\sqrt{2\beta + \nu^2} - \nu\right) \frac{\log(1+t)}{\sigma}\right) \\
&= (1+t)^{-\left(\sqrt{2\beta + \nu^2} - \nu\right)/\sigma}, \\
&=: \exp(-\Phi(t; \beta)).
\end{aligned} \tag{29}$$

where \mathbf{E}_1 is the expectation associated with Y when started from 1 and

$$\psi_\beta(x) = \exp\left(\left(\sqrt{2\beta + \nu^2} - \nu\right)x\right)$$

is the increasing fundamental solution for Brownian motion with drift (see [5] p.132). Notice that the additive process X is a deterministic time change of the first hit process of Brownian motion with drift, which is a subordinator. We use now Proposition 2.5 to study the moments of the perpetual integral functional

$$I_\infty = \int_0^\infty e^{-X_s} ds.$$

To simplify the notation (cf. (29)) introduce

$$\rho(\beta) := \left(\sqrt{2\beta + \nu^2} - \nu\right)/\sigma.$$

By formula (15) the n th moment is given by

$$\begin{aligned}
\mathbf{E}_1(I_\infty^n) &= n! \int_0^\infty dt_1 (1+t_1)^{-\left(\rho(n)-\rho(n-1)\right)} \int_{t_1}^\infty dt_2 (1+t_2)^{-\left(\rho(n-1)-\rho(n-2)\right)} \\
&\quad \times \int_{t_2}^\infty dt_3 \cdots \int_{t_{n-1}}^\infty dt_n (1+t_n)^{-\rho(1)} \\
&= \begin{cases} \frac{n!}{\prod_{k=1}^n (\rho(k) - k)}, & \text{if } n < n^*, \\ +\infty, & \text{if } n \geq n^*, \end{cases}
\end{aligned}$$

where

$$n^* := \min\{n \in \{1, 2, \dots\} : \rho(n) - n \leq 0\}.$$

Condition (16) in Corollary 2.6 takes in this case the form

$$\rho(n) - \rho(n-1) > 1. \tag{30}$$

This being a sufficient condition for the finiteness of $m_\infty^{(n)}$ we have

$$\rho(n) - \rho(n-1) > 1 \quad \Leftrightarrow \quad \rho(n) - n > 0. \quad (31)$$

Consider now the case $\nu = 0$. Then

$$\rho(n) > n \quad \Leftrightarrow \quad n < \frac{2}{\sigma^2}, \quad (32)$$

i.e., smaller the volatility (i.e. σ) more moments of I_∞ exist, as expected. Moreover, in this case

$$\begin{aligned} \rho(n) - \rho(n-1) > 1 &\quad \Leftrightarrow \quad \sqrt{2n} + \sqrt{2(n-1)} < \frac{2}{\sigma} \\ &\quad \Leftrightarrow \quad 2n - 1 + \sqrt{4n(n-1)} < \frac{2}{\sigma^2} \end{aligned} \quad (33)$$

showing, in particular, that when σ is “small” there exist ”many” n satisfying (32) but not (33).

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