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Explicit model predictive control for PDEs: The case of a heat equation

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Abstract: Explicit model predictive control design is carefully developed for discrete-time linear plants on Hilbert spaces, and we highlight the role of the so-called Slater condition in the reliable explicit solution of the MPC optimization. We then proceed to present an explicit MPC algorithm that accounts for the stabilization and input constraints satisfaction. We do structure preserving temporal discretization of the infinite-dimensional parabolic PDE system by application of the Cayley transformation. The salient feature of explicit MPC design is the realization of the region-free approach in explicit MPC design with identification of active constraint sets to realize optimal stabilization and constraints satisfaction. Finally, the resulting design is illustrated by the application to the PDE model given by an unstable heat equation with boundary actuation and Neumann boundary conditions. The example demonstrates simultaneous stabilization and input constraints satisfaction on the one hand, and on the ability to deal with a relatively high plant dimension and a long optimization horizon on the other hand.

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Keywords: Explicit MPC, Distributed Parameter Systems, Heat Equation, Boundary Control

1. INTRODUCTION

Model predictive control (MPC) designs permeate the field of the control algorithm realizations that account for important control related features, such as stabilization, optimality, constraint satisfaction and computational realizability. In particular, the fundamental issues of stabilization and constraint satisfaction in MPC design formulation for finite-dimensional systems have been explored in prior contributions, Rawlings et al. (2020), and other important aspects of nonlinear model predictive control have been explored in Grüne and Pannek (2017). However, the former contributions do not in detail address the important and fundamental aspects of model predictive control design realizations when models are given by distributed parameter systems.

Along the lines of various model predictive control designs, the important realization is given by the explicit linear quadratic regulator design for constrained linear systems Bemporad et al. (2002); Johansen et al. (2002). The advantage of solving the MPC optimization step explicitly (whenever this is possible) is computational speed completely superior to solving the MPC optimization implicitly. However, the problems arise in the realization of the explicit MPC control designs when the number of state variables increases, since this leads to an explosion of explicit controller complexity. Moreover, there are no further guidelines on how to address the setting when the state is of the infinite-dimensional nature as it is the case given by the PDE models.

The scientific contributions in the area of control of distributed parameter systems are quite mature and well established, see seminal contributions (Curtain and Pritchard (1978); Curtain and Zwart (1995); Krstic and Smyshlyaev (2008)). The most important aspects of prior contributions were given by synthesis and analysis of full state feedback control laws as well as output feedback designs with the emphasis on boundary/point based actuation and/or observation. Along the same lines, the optimal control strategies were explored which lead to the complex realization of the Riccati equations for the distributed parameter systems, see (Curtain and Zwart (1995); Ito and Kunisch (2002)). However, there is a lack of contributions that account for constrained input and/or state/output setting, optimality, and the infinite-dimensional DPS setting.

Motivated by the above, we develop an efficient explicit MPC algorithm based on the region-free approach to explicit MPC design; see the survey Kvasnica et al. (2019) for a nice introduction. Region-free explicit MPC allows us to evade almost all problems related to explicit MPC related to controller complexity, and degeneracy in the Karush-Kuhn-Tucker optimality conditions (Spjøtvold et al. (2006)). While the practical problems related to state

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space exploration, degeneracy in the KKT conditions, and storing the explicit control law in an efficiently searchable data structure has hampered the development of explicit MPC in general, the development of region-free explicit MPC has received little attention after the fundamental paper by Gupta et al. (2011), due to the lack of an efficient method to find the optimal active set.

There exists some theory on MPC for PDE systems, mainly developed by Kunisch and coworkers (Azmi and Kunisch (2016, 2018); Azmi et al. (2018); Azmi and Kunisch (2019)), but there the setting is mainly in continuous time and all forms of constraints have been excluded. In practical implementations, however, it is much easier to work in discrete time. We utilize the Cayley transformation that preserves the structural and energy preservation properties of the underlying PDE models, building on some prior contributions in the area of model predictive control designs applied to distributed parameter systems, see Dubljevic and Humaloja (2020). In particular, asymptotic stability is preserved under this transformation; see Curtain and Oostveen (1997); Havu and Malinen (2007); Hairer et al. (2006).

The rest of the paper is organized as follows. In §2, we present the heat equation that serves as an illustrative example throughout the paper, and in §3, we describe the explicit MPC design that fits PDE systems. The focus of §4 is the efficient and reliable solution of the optimization step. We provide a rudimentary but very fast algorithm for explicitly solving the optimization in the MPC controller. We test the algorithm in a numerical simulation in §5, where we treat an unstable heat equation with boundary actuation and Neumann boundary conditions. Finally, §6 contains some concluding remarks.

This paper is a companion paper of a full journal paper, which will soon be submitted for publication. This exposition here has been simplified and the proofs have been omitted for transparency. The PDE example in this paper also differs considerably from the full paper. Please find the full paper for proofs and a more detailed exposition.

2. SYSTEM MODEL

The motivating example belongs to the class of distributed parameter systems frequently encountered in the industrial engining process control practice. Namely, the heat conduction (diffusion) systems are omnipresent examples of the models that take the form of parabolic PDEs where natural imposed boundary conditions imply the cooling on one end as the boundary actuation element while the other end is considered to be isolated, see Fig. 1.

We consider a one-dimensional heat equation on the spatial interval $\xi \in [0,1]$ with Neumann boundary control at $\xi = 0$:

$$\frac{\partial}{\partial t}x(\xi,t) = \frac{\partial^2}{\partial \xi^2}x(\xi,t), \qquad x(\xi,0) = x_0(\xi) \tag{1a}$$

$$0 \equiv \left.\frac{\partial}{\partial \xi}x(\xi,t)\right|_{\xi=1} \tag{1b}$$

$$u(t) = \left.\frac{\partial}{\partial \xi}x(\xi,t)\right|_{\xi=0}. \tag{1c}$$

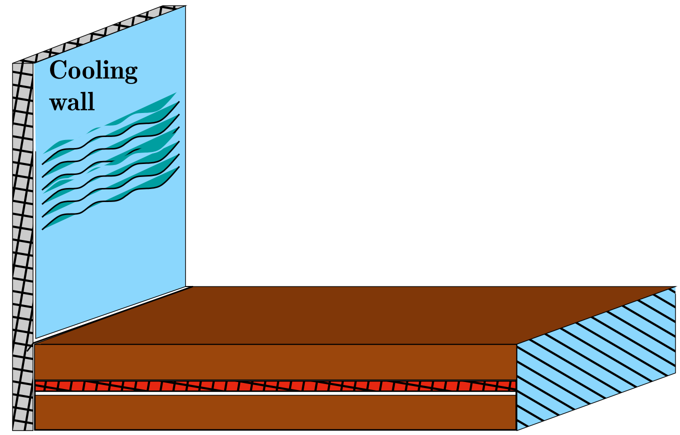

Fig. 1. Boundary heated slab with cooling on one side.

The above boundary control problem can be written as an abstract boundary control system (Tucsnak and Weiss, 2009, Sect. 10.1)

$$\dot{x}(t) = \mathfrak{A}x(t)$$
$$\mathfrak{B}x(t) = u(t),$$

on the state space $L^2(0,1;\mathbb{R})$, where $\mathfrak{A}x(t) = x''(t)$ with domain $D(\mathfrak{A}) = \left\{x \in H^2(0,1;\mathbb{R}) : x'(1) = 0\right\}$ and $\mathfrak{B}x(\cdot,t) = x'(0,t)$. The operator $\mathfrak{A}|_{\ker(\mathfrak{B})}$ is the generator of an analytic C_0-semigroup on $L^2(0,1;\mathbb{R})$.

The boundary control problem can be further written as an abstract Cauchy problem $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$ on $L^2(0,1;\mathbb{R})$, where $\mathcal{A} := \mathfrak{A}|_{\ker(\mathfrak{B})}$ and the operator $\mathcal{B}$ satisfies (Tucsnak and Weiss, 2009, Rem. 10.1.6)

$$\langle \mathfrak{A}x, z\rangle = \langle x, \mathcal{A}^*z\rangle + \langle \mathfrak{B}x, \mathcal{B}^*z\rangle \quad \forall x \in D(\mathfrak{A}), z \in D(\mathcal{A}^*).$$

Using integration by parts, we obtain $\mathcal{A}^* = \mathcal{A}$ with $D(\mathcal{A}^*) = D(\mathcal{A})$ which then yields $\langle \mathfrak{B}x, \mathcal{B}^*z\rangle = x'(0)z(0)$, i.e., $\mathcal{B}^*z = z(0)$ for all $z \in D(\mathcal{A}^*) = D(\mathcal{A})$, and hence $\mathcal{B}$ is the Dirac delta distribution at zero.

We proceed to find a spectral representation for the heat equation using the eigenvalues and the corresponding eigenvectors. The eigenvalue equation $\mathcal{A}x = sx$ gives $x''(\xi) = sx(\xi)$ which for $s = 0$ has the solution $x(\xi) = 1_{[0,1]} \in D(\mathcal{A})$. For $s \neq 0$, the solutions are of the form $x(\xi) = \alpha\cosh(\sqrt{s}\xi) + \beta\sinh(\sqrt{s}\xi)$. Since $x \in D(\mathcal{A})$, we have $x'(0) = 0$ which gives $\beta = 0$, and then $x'(1) = 0$ gives $\alpha\sinh(\sqrt{s}\xi) = 0$, i.e., $s = -(k\pi)^2$ for any $k \in \mathbb{N}$. Thus, the eigenvalues of $\mathcal{A}$ are $\lambda_k = -(k\pi)^2$ for $k \in \mathbb{N}_0$ and the corresponding normalized eigenvectors are $\phi_0 = 1$ and $\phi_k = \sqrt{2}\cos(k\pi\xi)$ for $k \in \mathbb{N}$.

Since the normalized eigenvectors of $\mathcal{A}$ form an orthonormal basis for $L^2(0,1;\mathbb{R})$, we can equivalently write $\mathcal{A}$ and $\mathcal{B}^*$ as (Tucsnak and Weiss, 2009, Sect. 2.6)

$$\mathcal{A}x = \sum_{k=0}^{\infty}\lambda_k \langle x, \phi_k\rangle \phi_k$$

$$\mathcal{B}^*x = \sum_{k=0}^{\infty}\langle x, \phi_k\rangle \mathcal{B}^*\phi_k = \sum_{k=0}^{\infty}\langle x, \phi_k\rangle \phi_k(0)$$

for $x \in D(\mathcal{A})$. Moreover, for any $s \in \rho(\mathcal{A})$, we have

$$(sI - \mathcal{A})^{-1}x = \sum_{k=0}^{\infty}\frac{1}{s-\lambda_k}\langle x, \phi_k\rangle \phi_k$$

which implies

$$\|\mathcal{B}^*(I-\mathcal{A})^{-1/2}\|^2 = \sum_{k=0}^{\infty} \frac{\phi_k(0)^2}{|1-\lambda_k|} < \sum_{k=0}^{\infty} \frac{2}{1+(k\pi)^2} = 1+\coth(1)$$

so that $\mathcal{B}^*$ is an admissible observation operator for $\mathcal{A}$ (Tucsnak and Weiss, 2009, Prop. 5.1.3). Since $\mathcal{A}$ is self-adjoint, $\mathcal{B}$ is equivalenty an admissible control operator for $\mathcal{A}$ (Tucsnak and Weiss, 2009, Thm. 4.4.3). Thus, the abstract Cauchy problem $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$ resulting from (1) is well-posed on $L^2(0,1;\mathbb{R})$.

We will next apply the Cayley transformation to map the continuous-time system $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$ into a discrete-time one $x_{k+1} = Ax_k + Bu_k$. For a given temporal discretization parameter $h > 0$, we define $\delta := 2/h$, and the Cayley transformation of $(\mathcal{A}, \mathcal{B})$ from continuous- to discrete-time is given by (Havu and Malinen, 2007, Sect. 1.3)

$$A = (\delta I + \mathcal{A})(\delta I - \mathcal{A})^{-1}, \qquad B = \sqrt{2\delta}(\delta I - \mathcal{A})^{-1}\mathcal{B}.$$

The Cayley transform follows from a Crank-Nicolson type time discretization (Havu and Malinen, 2007, Sect. 1.1)

$$\frac{x((k+1)h) - x(kh)}{h} \approx \mathcal{A}\frac{x((k+1)h) + x(kh)}{2} + \mathcal{B}u(kh)$$

after approximating $u(kh)$ by $u_k/\sqrt{h}$.

Using the above representations of $\mathcal{A}$ and $\mathcal{B}^*$, we obtain

$$Ax = \sum_{k=0}^{\infty} \frac{\delta+\lambda_k}{\delta-\lambda_k} \langle x, \phi_k\rangle \phi_k \tag{3}$$

and

$$\mathcal{B}^*(\delta I - \mathcal{A})^{-1}x = \sum_{k=0}^{\infty} \frac{\phi_k(0)}{\delta - \lambda_k} \langle x, \phi_k\rangle = \frac{B^*x}{\sqrt{2\delta}}$$

so that $\langle B^*x, u\rangle = \langle x, Bu\rangle$ gives

$$Bu = \sum_{k=0}^{\infty} \phi_k \frac{\sqrt{2\delta}}{\delta - \lambda_k} \phi_k(0)u. \tag{4}$$

Thus, for practical computations we can approximate A and B by restricting the spectral expressions (3) and (4) to a finite number of terms.

3. DESCRIPTION OF THE MPC ALGORITHM

We describe the MPC algorithm in a general framework where U and X are real Hilbert spaces, and A and B are bounded linear operators between these so that the discrete-time state equation

$$x_{n+1} = Ax_n + Bu_n, \qquad n \in \mathbb{N}_0, \tag{5}$$

makes sense. In case of the heat equation described in §2 we have $U = \mathbb{R}$, $X = L^2(0,1;\mathbb{R})$ and A, B are given by (3)–(4).

The objective is to drive x_n in (5) to zero in such a way that a quadratic cost functional

$$\sum_{n=0}^{\infty} \left(\langle Qx_n, x_n\rangle + \langle Ru_n, u_n\rangle\right), \tag{6}$$

is minimized while satisfying some affine constraints on the inputs u_n and the states x_n, which we here denote by $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right] \in \mathcal{W}$. Model predictive control (MPC), also called receding horizon control (RHC), achieves this objective by approximating the cost (6) by a finite sum. Choosing a *horizon* N and denoting $\mathbf{u}' := (u'_k)_{k=0}^{N-1}$, the input signal at time step n is chosen as $u_n = u_{*,0}$, where $\mathbf{u}_* = (u_{*,k})_{k=0}^{N-1}$ is the minimizer of the cost functional

$$J(\mathbf{u}', x_n) := \langle Px'_N, x'_N\rangle + \sum_{k=0}^{N-1} \left(\langle Qx'_k, x'_k\rangle + \langle Ru'_k, u'_k\rangle\right)$$

$$\begin{cases} x'_{k+1} = A'x'_k + B'u'_k, \\ x'_0 = x_n, \end{cases} \quad \begin{bmatrix} x'_k \\ u'_k \end{bmatrix} \in \mathcal{W}_k, \quad 0 \le k \le N-1, \tag{7}$$

where the weights $P = P^*, Q = Q^* \in \mathcal{L}(X)$ satisfy $P, Q \ge 0$ and $R = R^* \in \mathcal{L}(U)$ is *coercive*, i.e., it satisfies

$$\langle Ru, u\rangle \ge \varepsilon\|u\|^2, \qquad \forall u \in U \tag{8}$$

for some $\varepsilon > 0$ independent of u, and we assume that the *prediction model* in (7) is perfect, i.e., $(A', B') = (A, B)$. We further require that the constraints can be written in the affine form

$$\begin{bmatrix} x'_k \\ u'_k \end{bmatrix} \in \mathcal{W} \iff d - \mathcal{E}x'_k - Eu'_k \in [0,\infty)^p, \quad 0 \le k \le N-1, \tag{9}$$

with $d \in [0,\infty)^p$, $\mathcal{E} \in \mathcal{L}(X;\mathbb{R}^p)$, $E \in \mathcal{L}(U;\mathbb{R}^p)$ all given as part of the control problem.

We assume that the state-space can be decomposed as $X = X_u \oplus X_s$, corresponding to the unstable and stable eigenspaces of A, respectively, in such a way that X_u is finite-dimensional. In terms of the heat equation of §2, we have $X_u = \operatorname{span}\{\phi_0\}$ and $X_s = \overline{\operatorname{span}}\{\phi_k\}_{k=1}^{\infty}$. The operators A and B can be decomposed accordingly as $A = A_u \oplus A_s$ and $B = \left[\begin{smallmatrix} B_u \\ B_s \end{smallmatrix}\right]$, where A_u, A_s, B_u, B_s are obtained by restricting the spectral representations of A and B to the spanning vectors of the corresponding subspaces; from (3) we obtain, e.g., that

$$A_u x_u = \frac{\delta + \lambda_0}{\delta - \lambda_0} \langle x_u, \phi_0\rangle \phi_0.$$

To see how (7) approximates (6), consider the *dual-mode* controller whose control signal is of the form

$$\mathbf{u}' = (u'_k)_{k=0}^{N-1} \qquad \text{and} \qquad u'_k = 0, \quad k \ge N,$$

which switches to zero control after the control horizon. Since A may have finitely many unstable eigenmodes, we proceed as in Muske and Rawlings (1993) and impose a terminal constraint on the unstable modes to guarantee a finite terminal cost under the zero control. That is, we require that $(x_N)_u = 0$ which equivalently translates to an affine constraint

$$A_u^N x'_{0u} + \left[A_u^{N-1}B_u \; A_u^{N-2}B_u \; \ldots \; B_u\right]\mathbf{u}' = 0. \tag{10}$$

The feasibility of the terminal constraint (10) requires that the pair (A_u, B_u) is null-controllable in N time steps. With the unstable modes taken care of by the terminal constraint, we get from the strong stability of A_s that

$$\sum_{k=N}^{\infty} \left(\langle Qx_n, x_n\rangle + \langle Ru_n, u_n\rangle\right) = \langle Px_N, x_N\rangle,$$

where $P = 0_{X_u} \oplus P_s$ with P_s being the solution of the *Lyapunov equation*

$$A_s^* P_s A_s - P_s = -Q_s \tag{11}$$

on X_s; since A_s is strongly stable, (11) has a unique solution if and only if $Q_s^{1/2}$ is an *infinite-time admissible* *observation operator* for A_s (Curtain and Oostveen, 1997, Thm. 2.4.f). The latter always holds if $\mathcal{A}_s$ is exponentially stable (in continuous-time). In case of the heat equation of §2, $\mathcal{A}_s$ is exponentially stable and, e.g., for $Q = I_X$ the solution P_s is explicitly given by

$$P_s x_s = \sum_{n=0}^{\infty} (A_s^*)^n A_s^n x_s = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \left(\frac{\delta+\lambda_k}{\delta-\lambda_k}\right)^{2n} \langle x_s, \phi_k\rangle \phi_k.$$

Note that for a any fixed $k \in \mathbb{N}$, we get the sum of the geometric series

$$P_s\phi_k = \sum_{n=0}^{\infty} \left(\frac{\delta+\lambda_k}{\delta-\lambda_k}\right)^{2n} \phi_k = \frac{1}{1-r_k}\phi_k \tag{12}$$

where $r_k = \left(\frac{\delta+\lambda_k}{\delta-\lambda_k}\right)^2 = \left(\frac{\delta-(k\pi)^2}{\delta+(k\pi)^2}\right)^2$. However $r_k \to 1$ as $k \to \infty$.

4. EXPLICIT MPC DESIGN

As described in §3, nominal MPC solves a constrained quadratic optimization problem at every time step, in order to compute the next control action. *Explicit model predictive control* (here abbreviated to eMPC, not to be confused with *economic* MPC) was introduced in Johansen et al. (2002), Seron et al. (2000) and Bemporad et al. (2002). The point of eMPC is to express the optimal input $\mathbf{u}'$ sequence in (7) subject to the constraints (9) and (10) *explicitly* as a piecewise affine function of the parameter x_n, rather than as the implicit result of the constrained optimization problem described above.

Here we pursue a *region-free approach* to explicit MPC, which is much lighter than, and avoids some challenges of, the traditional explicit MPC approach. In particular, we do not need to pre-explore the state space, and we also circumvent storing the (often very complex) piecewise affine optimal control function.

Using a standard reformulation, see e.g. Seron et al. (2000), we can write the optimization problem (7) in the following form of a parametric quadratic program:

Solve

$$\operatorname{argmin}_{\mathbf{z}} \frac{1}{2}\langle H\mathbf{z}, \mathbf{z}\rangle$$

subject to

$$W' + S'x_n - G'\mathbf{z} \geq 0$$
$$A_u^N(x_n)_u + \left[A_u^{N-1}B_u \ \dots \ B_u\right](\mathbf{z} - H^{-1}Fx_n) = 0 \tag{13}$$

to obtain the minimizer $\mathbf{z}_*$, and then get the predicted optimal control sequence as $\mathbf{u}' = \mathbf{z}_* - H^{-1}Fx_n$.

In (13), the operators H, W', S', G' and F are given by the reformulation, and $H = H^*$ is coercive due to the assumed coercivity of R in (8). Using the coercivity of H and the closedness and convexity of the set of feasible $\mathbf{z}$, we can prove the following:

Theorem 1. Assume that the optimization problem (13) is *feasible*, i.e., that there is some $\mathbf{z}$ that satisfies all constraints. Then (13) has a unique solution $\mathbf{z}_*$.

Note that the equality constraint arising from the terminal constraint on the unstable eigenmodes can be equivalently written as two inequality constraints. Moreover, we can write $A_u^N(x_n)_u = \Gamma A^N x_n$ where Γ denotes the orthogonal projection onto X_u, so that W', S' and G' can be modified to account for the equality constraint in (13) by defining

$$W := \begin{bmatrix} W' \\ 0 \\ 0 \end{bmatrix}, \quad G := \begin{bmatrix} G' \\ \begin{bmatrix} I \\ -I \end{bmatrix} \left[A_u^{N-1}B_u \ \dots \ B_u\right] \end{bmatrix}$$
$$S := \begin{bmatrix} S' \\ \begin{bmatrix} I \\ -I \end{bmatrix} \left(\left[A_u^{N-1}B_u \ \dots \ B_u\right] H^{-1}F - \Gamma A^N\right) \end{bmatrix}. \tag{14}$$

The set of constraints that are active at the optimizer $\mathbf{z}_*$,

$$\mathbf{A} := \{k \mid f_k(\mathbf{z}_*) = 0, \quad k = 1, 2, \dots, \widetilde{p}\}$$

is referred to as the *optimal active set*, and its complement is $\mathbf{A}^c := \{k \mid k \notin \mathbf{A}, \quad k = 1, 2, \dots, \widetilde{p}\}$.

Let $G^{\mathbf{A}}$ denote the projection of the operator G onto the components indexed by $\mathbf{A}$. If $G^{\mathbf{A}}$ is surjective, then we say that the *Linear Independence Constraint Qualification (LICQ)* holds.

Using a Hilbert-space formulation of the Karush-Kuhn-Tucker theory in (Zeidler, 1985, §47.10), we can prove the following theorem which is a more precise formulation compared to what is presently found in the literature.

Theorem 2. Let $\emptyset \neq \mathbf{A} \subset \{1, 2, \dots, \widetilde{p}\}$ and let $\lambda := [\lambda_1 \ \dots \ \lambda_{\tilde{p}}]^*$ be a column vector of putative Lagrange multipliers. The following statements are true:

(1) Assume that $\mathbf{A}$, $x_n \in X \times U$, $\mathbf{z}_* \in U^N$ and $\lambda^{\mathbf{A}} \in \mathbb{R}^{\#\mathbf{A}}$ are such that LICQ holds and

$$W^{\mathbf{A}} + S^{\mathbf{A}}x_n = G^{\mathbf{A}}\mathbf{z}_*, \quad W^{\mathbf{A}^c} + S^{\mathbf{A}^c}x_n \geq G^{\mathbf{A}^c}\mathbf{z}_*,$$
$$\lambda^{\mathbf{A}} \geq 0, \quad H\mathbf{z}_* = -(G^{\mathbf{A}})^*\lambda^{\mathbf{A}}. \tag{15}$$

Then the unique minimizer of (13) is

$$\mathbf{z}_* = H^{-1}(G^{\mathbf{A}})^*\left(G^{\mathbf{A}}H^{-1}(G^{\mathbf{A}})^*\right)^{-1}(W^{\mathbf{A}} + S^{\mathbf{A}}x_n). \tag{16}$$

Moreover, the set of active constraints at $\mathbf{z}_*$ contains $\mathbf{A}$ and

$$\lambda^{\mathbf{A}} = -\left(G^{\mathbf{A}}H^{-1}(G^{\mathbf{A}})^*\right)^{-1}(W^{\mathbf{A}} + S^{\mathbf{A}}x_n). \tag{17}$$

Finally, *complementarity*, holds, i.e., $\lambda^{\mathbf{A}^c} = 0$.

(2) Conversely, assume that the parameter $x_n \in X \times U$ is such that the *Slater condition* holds, i.e., there is a $\mathbf{z} \in U^N$, such that $W + Sx_n - G\mathbf{z} \in (0,\infty)^{\tilde{p}}$.

If $\mathbf{z}_*$ is the minimizer of (13), and if $\mathbf{A}$ is the set of constraints active at $\mathbf{z}_*$, then (15) holds, together with $W^{\mathbf{A}^c} + S^{\mathbf{A}^c}x_n > G^{\mathbf{A}^c}\mathbf{z}_*$.

The Slater condition trivially implies feasibility (since the feasible set contains all the Slater points $\mathbf{z}$) and then Lemma 1 guarantees the existence of the minimizer $\mathbf{z}_*$. If the Slater condition does not hold, but the minimization (13) is nevertheless feasible, then the unique minimizer $\mathbf{z}_*$ still exists by Lemma 1, but then we have no guarantees that $\mathbf{z}_*$ (or $\lambda^{\mathbf{A}}$) has the affine representation (16) (or (17)), even if LICQ holds. The Slater condition is needed already in finite dimensions, see (Zeidler, 1985, pp. 56–57), but it is usually overlooked in the explicit MPC literature.

Even if item 2 in Theorem 2 is the part that is usually mentioned in the explicit MPC literature, in fact item 1 is also frequently used implicitly, and indeed item 1 is the one that we base our explicit MPC algorithm on in this paper. For constraints not involving the state, the Slater condition often holds for all parameter values x_n, but unfortunately *for us the Slater condition never holds*, due to the terminal constraint which is always active in (13).

We call an $\mathbf{A}$ satisfying (15) a *sufficient active set*, since it may be strictly smaller than the actual set of constraints active at the optimum $\mathbf{z}_*$, but such an $\mathbf{A}$ is nevertheless sufficient for guaranteeing optimality. Theorem 2 says nothing for $\mathbf{A} = \emptyset$, but it is clear that the optimizer is $\mathbf{z}_* = 0$ in this case (provided that the problem is feasible), and $\lambda = 0$ works as associated Lagrange multiplier.

We end this section by describing a simple algorithm the purpose of which is to find a sufficient active set $\mathbf{A}(x_n)$ for the parameter x_n. The algorithm is a kind of point location algorithm, which is concerned with active sets rather than with the more traditional critical regions. It is based on the idea that violated constraints should be activated and negative Lagrange multipliers should be locked to zero by deactivating the corresponding constraint, thus making use of complementarity. Moreover, at time step n, the sufficient active set from time $n-1$ is likely to be a fairly good warm start, as it is not expected that we cross too many critical regions in one step; see the idea of *facet flipping* in Tøndel et al. (2003).

Algorithm 1: Rudimentary search for a sufficient active set $\mathbf{A}$ and the corresponding optimizer $\mathbf{z}_*$. At the first time step $n = 0$, Alg. 1 is called with $\mathbf{A}' := \emptyset$. At subsequent time steps n, the warm start is the sufficient active set from the previous time step, $\mathbf{A}'_n = \mathbf{A}_{n-1}$.

```
Data: x_n, H^{-1}, F, G, S, W and warm start A'
Result: optimizer z_* and sufficient active set A
Calculate λ^{A'} using (17) with k = 0;
if some λ_k^{A'} < 0 then
    remove k with the most negative λ_k^{A'} from A';
else
    calculate z_* using (16);
    if some constraint is violated then
        add k with the most negative
          W^{k} + S^{k} x_n − G^{k} z_* to A';
    else
        return z_* and A := A';
        terminate;
return z_* and A as the result of this algorithm with
 the new warm start A'
```

Note that the algorithm only adds violated constraints to $\mathbf{A}'$, and that a redundant constraint is never violated. Hence, the only way that the algorithm can run into LICQ violation, is if a candidate active set $\mathbf{A}'$ is infeasible. With the constraints described in this paper, feasibility can be guaranteed by choosing the horizon N long enough.

It turns out that the simple and pragmatic active set algorithm in Alg. 1 allows very fast explicit MPC for our example, as we shall see in §5 below. The algorithm also highlights a fundamental difference between a sufficient active set and the full set of constraints active at the optimum: If $\mathbf{A}$ denotes the full optimal active set at an optimum $\mathbf{z}_*$, then $G^{\mathbf{A}}$ always contains the last two lines of G, by (14), and hence $G^{\mathbf{A}}$ always violates LICQ. However, the sufficient active sets found by Alg. 1 never violate LICQ, and hence they are always a proper subset of the full optimal active set. (All constraints in a sufficient active set $\mathbf{A}$ are active at the optimum $\mathbf{z}_*$, as a consequence of (16).)

5. TESTING THE ALGORITHM ON THE HEAT EQUATION

We demonstrate the proposed explicit MPC design with a numerical simulation of the heat equation described in §2. For the simulation, we choose the weights in the cost function (7) as $Q = I_X, R = 1$. We consider the cooling problem of a nonnegative temperature profile distribution, so that in continuous-time the input constraints are given by $u(t) \in [-3, 0]$ for all $t \geq 0$, which for the Cayley transform (Havu and Malinen, 2007, Sect. 1.1) correspond to $u_k \in \sqrt{h}\,[-3, 0]$ for all $k \in \mathbb{N}$ for the given discretization parameter h. In the simulation, we use $h = 1/40$ so that $\delta = 80$. We will then have to choose the prediction horizon N sufficiently large so that the constraint on the unstable subspace $A_u^N x_{0u} + \left[A_u^{N-1} B_u \ \ldots \ A_u B_u \ B_u\right] \mathbf{u} = 0$ can be satisfied within the input constraints. Note that the feasibility of this constraint has to be checked only at the first time step as $A_u = 1, B_u > 0$ and $u_k \leq 0$ so that $(x_{k+1})_u = (x_k)_u + B_u u_k \leq (x_k)_u$ for all $k \in \mathbb{N}_0$. A sufficient choice for the horizon length is $N = 7$.

The initial condition for the simulation is given as $x_0(\xi) = 3\xi^2 - 2\xi^3$. The evolution of the state profile under the explicit MPC controls with prediction horizon $N = 7$ is shown in Fig. 2. For computing the controls, the expressions of A, B and P_s are approximated from (3), (4) and (12), respectively, by using the first terms in the series up to $k = 15$. The discrete-time controls are converted to continuous-time using zero-order hold, i.e., $u(t) = u_k/\sqrt{h}$ for $t \in [kh, (k+1)h)$. Figure 2 shows that the state profile converges to zero in about one second under these controls.

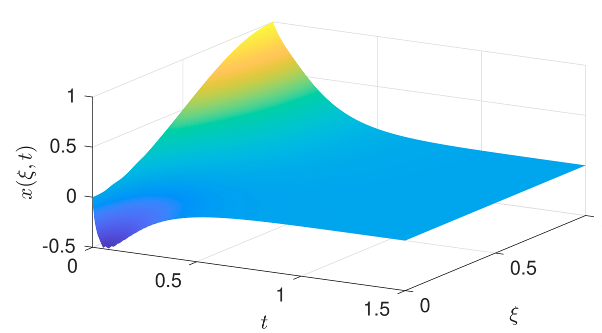

Fig. 2. State profile of the heat equation for $t \in [0, 1.5]$ under the explicit MPC law for $N = 7$.

The explicit MPC law that controls the state in Fig. 2 is shown in Fig. 3. For comparison, Fig. 3 also shows the explicit MPC law for the horizon length $N = 21$. This shows that the shorter horizon $N = 7$ has to sacrifice some optimality in order to satisfy the terminal constraint on the unstable subspace, whereas the longer horizon $N = 21$ achieves stabilization with less control effort. However, the discrete-time control costs are $J_7 \approx 2.89$ and $J_{21} \approx 2.77$ so that the benefit of tripling the prediction horizon is only minor. Amazingly, tripling the prediction horizion only increases the computational time by about 50 percent which is still only around 0.11 milliseconds per time step in Matlab on a 1.4 GHz Intel i5 processor.

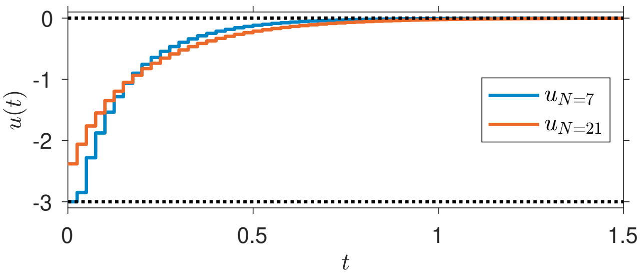

Fig. 3. The explicit MPC law for $N = 7$ and $N = 21$ (solid lines) along with the input constraints (dotted lines).

6. CONCLUSIONS

We extended explicit model predictive control to the distributed parameter plant setting and considered a parabolic PDE with boundary applied actuation. Based on a carefully formulated KKT theorem for the Hilbert-space setting, we provided a rudimentary algorithm fast enough for solving high-dimensional problems with a long MPC horizon. The role of the Slater condition in the explicit solution, and the difference between necessary and sufficient conditions for optimality were highlighted.

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