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Abstract. The falling rod paradox, i.e. the fact that the tip of an almost horizontal rod falls with an acceleration "higher than g", when the other end is hinged or supported, is a popular physics demonstration. It can be visualized by placing e.g. a coin on the tip of the rod and fixing a cup next to the coin. When the rod is released the free-falling coin is left behind and, if the vertical projection of its position is well aimed, the coin ends up in the cup. In this work we aim at visualizing the falling-rod paradox using a high speed camera and experimentally determine the point on the rod where the acceleration $a = g$ for various setups. To enhance the visualization we use evenly spaced beads on top of the rod. The experiments can be performed by pupils in the upper secondary school. Theoretical understanding may require first-year university classes in mechanics.

1. Introduction

The falling-rod (or -stick) paradox, i.e. the fact that the tip of an approximately horizontal rod hinged at the other end accelerates with $a = 3g/2$ has been discussed in several publications over the years.[1, 2, 3, 4, 5, 6] There isn't really anything paradoxical about the rod. As one end is hinged, the rod is not in free fall and the description of the dynamics is entirely governed by Euler's (torque) law applied to the hinged end:

$$\tau = mg\frac{L}{2}\sin\theta = I\alpha, \tag{1}$$

where τ is the torque due the weight mg of the rod, L the length of the rod, $I = mL^2/3$ the moment of inertia around the hinge, θ the angle between the rod and the gravitational force on the rod, and α the angular acceleration. For a homogeneous straight rod $\alpha = 3g/(2L)$ when the rod is horizontal and hence the tip accelerates with $a = \alpha L = 3g/2$. The fact that the tip of the rod is forced to accelerate "faster than g" has well-known consequences: Falling chimneys usually spontaneously break into two parts[4] and falling trees bend upwards when cut down. Also the coin-on-the-tip-of-rod demonstration, described in the abstract (and mentioned in some of the references), is a popular class-room example frequently employed. The acceleration of the coin on the falling rods has also been recorded using high-speed video cameras.[7] The exact dynamics of the falling rod involves elliptic integrals and is therefore beyond the scope of the upper-secondary curriculum. Adaptation and approximations to make the experiment accessible to pre-university curricula has been discussed.[5] With the development of high-speed cameras and mobile phones with good-quality high-frame rate cameras experiments in dynamics are nowadays readily accessible to pupils.

In this work we focus on visualizing the acceleration of a falling rod by placing evenly spaced beads on the falling rod and recording the fall with a high-speed video camera. The experiment can be performed as a class-room demonstration on upper-secondary level and the physics presented is also accessible to the pupils, provided the school syllabus includes the concepts of torque, moment of inertia and rotational dynamics. Similar experiments with weights attached to falling rods have been used for stimulating students' physical thinking.[8]

2. Experimental

The experimental setup was the following: A steel rod of length $L = 1.000 \pm 0.005$ m and mass $m = 306,3 \pm 0,6$ g with 40 evenly spaced ($r_0 = 2.50 \pm 0.05$ cm) premanufactured holes was placed on a wooden block at an angle of $\sim 35°$ with the horizontal. A nail prevented unwanted horizontal movement and acted as a simple hinge for the end of the rod resting on the block (Fig. 1). Identical non-magnetized steel beads from a ball bearing were placed on the rod prior to its release. The beads had a diameter of 10,0 mm and a mass of $4,08 \pm 0,1$ g. An additional mass of 1,0 kg was placed, in some of the experiments, on the rod in order to vary the acceleration of the tip. For recording the

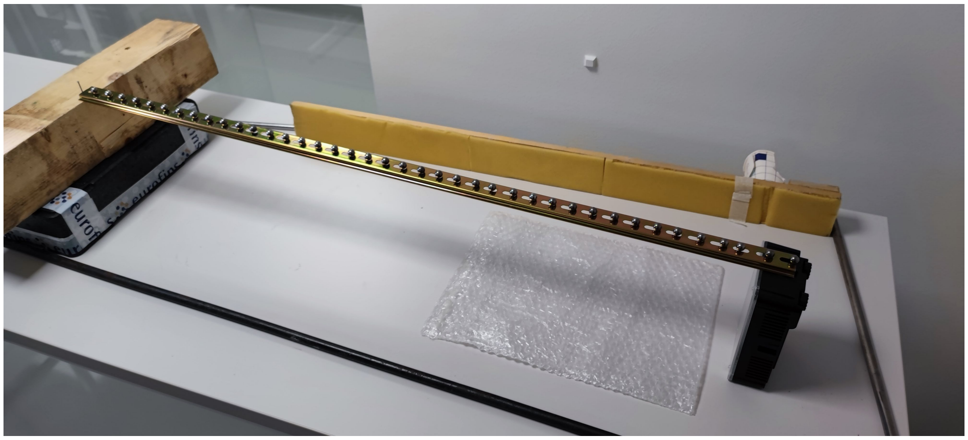

Figure 1. The steel rod used in the experiments. The rod is resting on a wooden block and a nail prevents the rod from moving laterally on the block. Prior to release, the beads were placed in the oval holes and moved towards edges facing the hinged-end of the rod.

falling rod a Photron FASTCAM SA3 120K black-and-white high-speed video camera was used at a frame rate set at 1000 fps. From the video footage, the "$a = g$ point" was readily determined by stopping at the frame in which the rod was horizontal. Beyond that point the steel beads become airborne at positions defined by the initial inclination of the rod, whereas beads closer to the hinge remain resting on the rod. Owing to the elevation provided by the wood block the rod could continue its fall below the horizontal level. For quantitative analysis, the known length of the rod was used for calibrating distance measurements in the video footage.

3. Theory

Although the basic physics of the falling rod is described by Eq. 1 there are some subtleties in the experiment outlined above. Due to the steel beads the falling rod and the beads can no longer be consider forming a solid body. Indeed, when beads near the free end become airborne the combined center of mass of the rod and the beads is shifted towards the hinge. Furthermore, the support force between the rod and remaining beads vary depending on how close or far the bead is from the hinge, due to varying linear acceleration of the rod. Let us begin by discussing what happens if an additional weight is attached to the rod. The problem has been discussed before.[9] Upon introducing a point-like mass M at the distance r from the hinge, Eq. 1 becomes:

$$(Mr + m\frac{L}{2})g\sin\theta = I'\alpha, \tag{2}$$

where $I' = Mr^2 + mL^2/3$ is the modified moment of inertia. Solving for α we obtain

$$\alpha = g\sin\theta\frac{mL/2 + Mr}{mL^2/3 + Mr^2}.$$

The value for α can be greater or smaller than $3g/2L$ depending on where the point-like mass M is placed. It is also interesting to maximize α by varying r, as discussed e.g. in Refs [10] and [11]. By taking the derivative of α with respect to r and treating all other quantities as constants the r value that gives the largest angular acceleration is obtained as

$$r = \frac{(m^2/4 + mM/3)^{1/2} - m/2}{M} L. \tag{3}$$

Using similar reasoning it is rather straight forward to show that if two equal point-like masses are fixed to the rod their combined effect cancel if their positions r_1 and r_2 fulfill the equation

$$\frac{3}{2} r_1^2 - L r_1 = L r_2 - \frac{3}{2} r_2^2. \tag{4}$$

For all real solutions with r_1 and r_2 located on the rod, $\alpha = 3g/(2L)$ always holds.

Next, we will explore the effect of putting the steel beads on the rod. We examine the rod when it has reached the horizontal position. If the ith bead remains at rest on the rod, its dynamics is readily obtained from Newton's second law as

$$m_{\mathrm{b}} g - N_i = \alpha r_i m_{\mathrm{b}}, \tag{5}$$

where m_{b} is the mass of the bead, r_i the distance to the hinge and N_i the support force between the rod and the bead. As the rod is horizontal all motion is momentarily along the vertical direction. The distance r_i can be given as $r_i = (i-1)r_0 + r^*$, where r_0 is the distance between adjacent beads and r^* is the distance from the first hole to the hinged end. The total torque acting on the rod due the support forces from the beads remaining on the rod and from the weight of the rod itself casts Eq. 1 into the following form:

$$\tau = mgL/2 + \sum_{i=1}^{n'} (m_{\mathrm{b}} g - m_{\mathrm{b}} \alpha r_i) r_i = \frac{1}{3} m L^2 \alpha, \tag{6}$$

where n' indicates the last bead for which the support force $N_i \geq 0$. Upon solving Eq. 6 for α we get:

$$\alpha = g \frac{mL/2 + m_{\mathrm{b}} \sum_{i=1}^{n'} r_i}{\frac{1}{3} m L^2 + m_{\mathrm{b}} \sum_{i=1}^{n'} r_i^2}. \tag{7}$$

This is really a recursive equation as the upper limit of the sum n' depends on α, through the support force in Eq. 5. Upon Setting $N_i = 0$, $r_i = (n'-1)r_0 + r^*$ and solving for n' we get $n' = g/(\alpha r_0)$, which is not necessarily an integer, but for all $i \geq n'$ the support force N_i is zero and the corresponding beads airborne.

4. Results and discussion

Several rounds of experiments were done with the rod either partially or fully loaded with a maximum of 40 beads. In the first round the rod was released with 40 beads from the angle of $\sim 35°$ and it was found that 14 beads were airborne at the moment when the rod reached the horizontal level (Fig. 2). Students familiar with the $y \propto t^2$ dependence of falling objects, might be astonished to see that the airborne beads are linearly aligned and not forming a parabola. However, the beads start linearly aligned on the rod and loose contact with the rod immediately after its release and subsequently accelerate with g. Therefore, they travel equal distances in equal times and thus retain the initial linear alignment. Using a ruler, the "$a = g$ point" can be estimated from a print-out

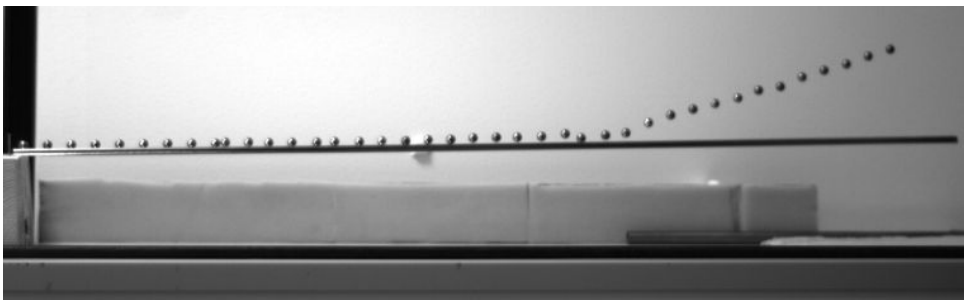

Figure 2. The falling rod loaded with 40 steel beads as it reaches the horizontal level.

of Fig. 2 as the vertex point formed between the airborne and resting beads, and it is found to be 38 cm from the free end. Alternatively, the point can be determined rather exactly by determining the intersection of the lines fitted to the airborne beads and the resting beads. The result was that the "$a = g$ point" is located at 37.5 ± 0.3 cm from the free end. Theoretically, for a rod without the beads, the point should be located at exactly one third or 33.33 cm from the free end. Considering that air resistance and minute frictional losses at the "hinge" should slow down the rod a bit, a result that gives a larger acceleration than predicted by theory is remarkable. However, as outlined by Eqs. (2)-(7) and discussed in the next paragraph the inner beads give rise to an additional torque that speeds up the rod. Therefore, in the second round only 15 beads were placed on the rod, starting from the free end, resulting in 13 airborne beads when the experiment was repeated (Fig 3). In this case the "$a = g$ point" was obtained at

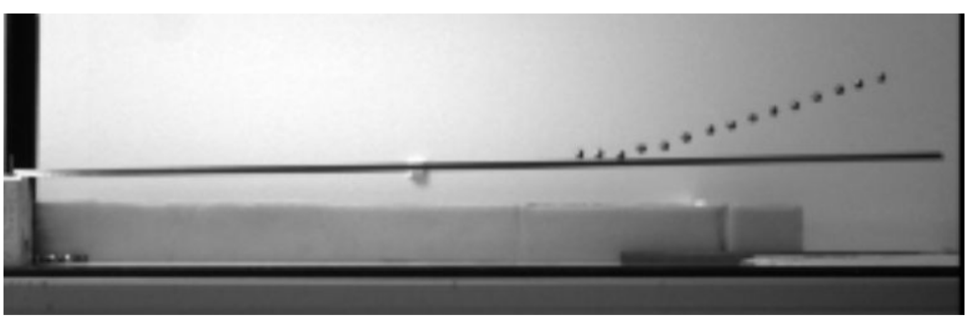

Figure 3. The falling rod loaded with 15 steel beads as it reaches the horizontal level.

34.6 cm from the free end using a ruler or at 33.3 ± 0.3 cm, when using the intersecting lines. The latter value is in excellent agreement with theory for the the unloaded rod, indicating that the effect of the two beads remaining on the rod had negligible effect on the falling rod.

Returning to the experiment with 40 beads on the rod, we can check if the "$a = g$ point" located at 37.5 cm is consistent with Eq. (7), that includes the additional accelerating effect of the beads on the rod. The number of beads still resting on the rod in Fig. 2 is $n' = 26$. The "$a = g$ point" is obtained as $L - L\alpha/g$, where α is numerically obtained from the equation, yielding 37.62 cm, which is very close to the experimental value. This would indicate that losses from air resistance and friction in the "hinge" are negligible in the experiments.

As our experiment is basically a crude accelerometer, we were also interested in speeding up the falling rod, with a well-defined additional weight. For that purpose a mass $M = 1,0$ kg was attached to the rod. Upon inserting the values for M and m into Eq. (3) the optimum distance maximizing α was found to be $r = 20.1$ cm, giving rise to a theoretical angular acceleration of $\alpha = 2.4852g/L$ and an "$a = g$ point" located at 59.75 cm from the free end. In this case, 23.3 beads should be airborne, when counting from the free end, assuming negligible effect of the beads on the acceleration of the rod. Fig (4) shows the corresponding experiment conducted with 26 steel beads, of which 24 are airborne. From the figure, the "$a = g$ point" is interpolated to 61.3 ± 0.3 cm

Figure 4. The falling rod loaded with 26 steel beads and a fixed weight of 1,0 kg, located 20.1 cm from the hinged end, as it reaches the horizontal level.

from the free end. This value slightly exceeds the theoretical value. The theoretical value can be "improved" by considering not only the weight M but also the two beads resting on the rod. This increases the theoretical value very slightly to 59.78 cm for the "$a = g$ point", and is still smaller than the experimentally obtained. However, when an additional heavy weight M is used the rod bends a little prior to the release and the potential energy stored as tension in the bent rod may give a slight jerk to the free end during the release. This could increase the initial acceleration of the free end, which could be the reason for obtaining a value slightly exceeding the theoretical one.

For the sake of argument, we attached the mass M to the free end of the rod leading to an almost null result with only one bead being airborne as expected (Fig. 5). We also tested filming the rod with the camera of an ordinary "smart" phone using a frame rate of 60 fps (Fig 6.), which produced similar results as described above. Therefore, the experiment can be easily performed in a class room without the need of any special high-frame-rate equipment. However, with a much lower frame rate it is not possible to

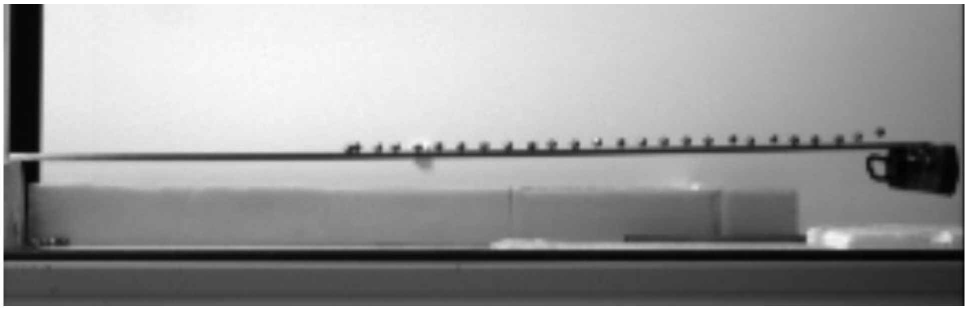

Figure 5. The falling rod loaded with 26 steel beads and a fixed weight of 1,0 kg, located at the free end, as it reaches the horizontal level.

freeze the motion of the rod nor the beads. A brighter light source could perhaps reduce the shutter speed and reduce motional blurring. On the other hand, the blurring of the rod nicely illustrates the increase in linear velocity for points closer to the free end.

Figure 6. The falling rod loaded with 15 steel beads as it reaches the horizontal level. Frame extracted from a mobile phone filming at 60 fps the same experiment as in Fig. 3.

5. Conclusions

By putting equally spaced steel beads on a falling rod, the "$a = g$ point" is easily visualized using a high-speed video camera. By inserting weights at various positions on the rod, the acceleration of the free end of the rod can be explored. Inserted weights affect the dynamics of the rod both by adding to the total torque and the moment of inertia, yielding results that might be hard for the students to predict before doing the experiments as the torque grows linearly with the distance to the hinge and the moment of inertia quadratically.

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