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THE BLOCH SPACE ON THE UNIT BALL OF A HILBERT SPACE: MAXIMALITY AND MULTIPLIERS

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Abstract. We prove that, as in the finite dimensional case, the space of Bloch functions on the unit ball of a Hilbert space contains under very mild conditions any semi-Banach space of analytic functions *invariant* under automorphisms. The multipliers for such Bloch space are characterized and some of their spectral properties are described.

1. Introduction and preliminaries

All over $(X, \|\cdot\|)$ denotes a semi-Banach space of analytic functions on the unit ball of a Hilbert space H that is *invariant* under automorphisms φ of the ball B_H in the sense that for all $f \in X$, we have

$$f \circ \varphi \in X \text{ and } \|f \circ \varphi\| = \|f\|.$$

A function $f : B_H \to \mathbb{C}$ is said to be a Bloch function [2] if

$$\|f\|_{\mathcal{B}} := \sup_{x \in B_H} (1 - \|x\|^2)\|\nabla f(x)\| < \infty.$$

By $\mathcal{B}(B_H)$, we denote the space of Bloch functions defined on B_H. We will consider besides the semi-norm $\|\cdot\|_{\mathcal{B}}$, the semi-norm

$$\|f\|_{inv} := \sup_{x \in B_H} \|\widetilde{\nabla} f(x)\| < \infty$$

where the *invariant gradient* $\widetilde{\nabla} f$ is defined by $\widetilde{\nabla} f(a) = \nabla(f \circ \varphi_a)(0)$ for any $a \in B_H$ (see below the definition of the automorphism φ_a.) Both semi-norms were shown to be equivalent [2, Theorem 3.8] and render $\mathcal{B}(B_H)$ a semi-Banach space. The latter semi-norm $\|f\|_{inv}$ is invariant under automorphisms. So $\mathcal{B}(B_H)$ is invariant under automorphisms of B_H. Associated to these semi-norms there are the corresponding (equivalent) norms,

$$\|f\| := |f(0)| + \|f\|_{\mathcal{B}} < \infty, \text{ and } |||f||| := |f(0)| + \|f\|_{inv}.$$

In this note it is proved that any other invariant space $(X, \|\cdot\|)$ possesing a nontrivial linear functional continuous for the compact open topology is continuously embedded in $\mathcal{B}(B_H)$. This was already proved in 1982 by R. Timoney [7] for finite dimensional Hilbert spaces and recalled in K. Zhu's book [9], whose proof inspired strongly ours, in spite that there is a missing assumption in the result's statement. Also the multipliers of the Bloch space are characterized

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and the invertibility, spectrum and essential spectrum of the linear operators they give raise to are described.

A summary about automorphisms of the unit ball B_H follows: The analogues of Möbius transformations on H are the mappings $\varphi_a : B_H \to B_H$, $a \in B_H$, defined according to

$$\varphi_a(x) = (s_a Q_a + P_a)(m_a(x)) \tag{1}$$

where $s_a = \sqrt{1 - \|a\|^2}$, $m_a : B_H \to B_H$ is the analytic map

$$m_a(x) = \frac{a - x}{1 - < x, a >}, \tag{2}$$

$P_a : H \to H$ is the orthogonal projection along the one-dimensional subspace spanned by a, that is,

$$P_a(x) = \frac{\langle x, a\rangle}{\langle a, a\rangle} a$$

and $Q_a : H \to H$, is its orthogonal complement, $Q_a = Id - P_a$. Recall that P_a and Q_a are self-adjoint operators since they are projections, so $\langle P_a(x), y\rangle = \langle x, P_a(y)\rangle$ and $\langle Q_a(x), y\rangle = \langle x, Q_a(y)\rangle$ for any $x, y \in H$.

The automorphisms of B_H turn to be compositions of such analogous Möbius transformations with unitary transformations U of H, that is, self-maps of H satisfying $\langle U(x), U(y)\rangle = \langle x, y\rangle$ for all $x, y \in H$.

The pseudo-hyperbolic and hyperbolic metrics on B_H are respectively defined by

$$\rho_H(x, y) := \|\varphi_x(y)\| \quad \text{and} \quad \beta_H(x, y) := \frac{1}{2} \log \frac{1 + \rho_H(x, y)}{1 - \rho_H(x, y)}.$$

By $\mathcal{H}(B_H)$ we denote the space of complex-valued analytic functions on B_H, and by $H^\infty(B_H)$ the subspace of $\mathcal{H}(B_H)$ of bounded functions endowed with the norm, denoted $\|\cdot\|_\infty$, of uniform convergence on B_H. It is known that $H^\infty(B_H) \subset \mathcal{B}(B_H)$ with continuous inclusion [2]. For background on analytic functions we refer to [8].

2. Maximality

Before proving our main result Theorem 2.1, we show some others that we need.

Lemma 2.1. *(a) Every term in the Taylor series of $f \in X$, belongs to X as well.*
(b) If there is a non-constant function $g \in X$, then every linear continuous functional on H lies in X.

Proof. (a) Recall that for $f \in X$, the m-homogeneous term in its Taylor series at 0 of f is given by $f_m(z) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(\lambda z)}{\lambda^{m+1}} d\lambda$. By putting $\lambda = e^{i\theta}$, it turns out that $f_m(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta} z)}{e^{im\theta}} d\theta$. And this function belongs to X because the suitable Riemann sums are functions in X since $z \rightsquigarrow e^{i\theta} z$ are automorphisms of B_H.

(b) Recall also that $f_1(z) =< z, \overline{\nabla f(0)} >$. Suppose $\nabla f(0) = 0$ for all $f \in X$. This means that $\nabla f(z) = 0$ for all $f \in X$: indeed for all automorphisms φ_a, we have

$$d(f \circ \varphi_a)(a)(w) = \big(df(0) \circ d\varphi_a(a)\big)(w) =< d\varphi_a(a)(w), \overline{\nabla f(0)} >= 0.$$

Thus, $\nabla(f\circ\varphi_a)(a)=0$, for all $f\in X$. And since $\varphi_a\circ\varphi_a=id$, we get

$$\nabla f(a)=\nabla\big((f\circ\varphi_a)\circ\varphi_a\big)(a)=0.$$

Consequently, every $f\in X$ is constant. This is in contrary to the assumption, so there is a norm one linear functional υ in H which belongs to X.

Since the dual space of a Hilbert space is also a Hilbert space, any norm one linear functional in H can be obtained by composing υ with a suitable unitary transformation that is, of course, an automorphism of the ball. □

Recall that a (continuous) finite type polynomial on a normed space E is a linear combination of powers of functionals in the dual space E^*.

Proposition 2.1. *If there is a non-constant function $g\in X$, then all finite polynomials lie in X.*

Proof. According to Lemma 2.1 (b), for each $a\in B_H$, the linear functional $a^*(z)=<z,a>$ is an element of X. Thus $a^*\circ\varphi_a\in X$. Since

$$(3)\qquad \begin{array}{c}(a^*\circ\varphi_a)(z)=P_a\Big(\frac{a-z}{1-<z,a>}\Big)=\Big\langle\frac{a-z}{1-<z,a>},a\Big\rangle=\\ \frac{\|a\|^2-<z,a>}{1-<z,a>}=\|a\|^2+\sum_{n=0}^{\infty}(\|a\|^2-1)<z,a>^{n+1},\end{array}$$

we apply Lemma 2.1 (a) to assure that the powers of the linear functional a^* are in X. Hence the finite type polynomials belong to X. □

Theorem 2.1. *Assume that in the semi-Banach space X there is a nonconstant function and that there is a nonzero linear functional L on X that is continuous for the compact open topology τ_0. Then $X\subset\mathcal{B}(B_H)$. If further $L(1)\neq 0$, then $X\subset H^\infty(B_H)$.*

Proof. We first assume that $L(1)=0$. Let e be a vector in some orthonormal basis of H. We show by contradiction that there exists an automorphism φ of B_H such that for the linear functional $L_\varphi:=L\circ C_\varphi$, one has $L_\varphi(e^*)\neq 0$, where C_φ is the composition operator given by right composition with φ.

So assume that $L_\varphi(e^*)=0$ for all automorphisms φ of B_H. Set $a=re,\ 0<r<1$, and consider the Taylor series of $e^*\circ\varphi_a=\frac{1}{r}a^*\circ\varphi_a$ obtained from (3):

$$e^*\circ\varphi_a(z)=r+\sum_{n=0}^{\infty}\frac{(r^2-1)}{r}<z,a>^{n+1}=r+(r^2-1)\sum_{n=0}^{\infty}r^n<z,e>^{n+1}.$$

Since this series is τ_0 convergent, we have for all $0<r<1$,

$$0=L_{\varphi_a}(e^*)=L(e^*\circ\varphi_a)=rL(1)+(r^2-1)\sum_{n=0}^{\infty}r^nL(<\cdot,e>^{n+1}).$$

Hence $L((e^*)^k)=L(<\cdot,e>^k)=0$ for all $k\in\mathbb{N}$.

For any other element $w\in H,\ w\neq 0,\ \|w\|=1$, there is an isometric isomorphism- and also an automorphism of the ball- ψ of H exchanging e and w, so for all automorphisms φ, $L_\varphi(w^*)=L_\varphi(e^*\circ\psi)=L_{\psi\circ\varphi}(e^*)=0$. Therefore we argue as in the paragraph above to conclude that $L((w^*)^k)=0$ for all $k\in\mathbb{N}$.

Then by linearity, $L(P) = 0$, for all finite type polynomials P in H. And bearing in mind that the finite type polynomials are τ_0-dense in the space $\mathcal{H}(B_H)$ ([8, 28.1 Theorem]), then $L(f) = 0$ for all $f \in X$. This is a contradiction.

Therefore we may assume since all L_φ are also τ_0-continuous that $L(e^*) \neq 0$.

Let $f \in X$. For any compact subset $K \subset B_H$, the series $\sum_{m=0}^{\infty} e^{imt} f_m(z)$ is uniformly convergent in $[0, 2\pi] \times K$ (use Cauchy inequalities). Thus the series $\sum_{m=0}^{\infty} e^{imt} f_m$ is τ_0-convergent to $f(e^{it}\cdot)$. So, $L(f(e^{it}\cdot)) = \sum_{m=0}^{\infty} e^{imt} L(f_m)$, and

$$\Big|\sum_{m=0}^{\infty} e^{imt} L(f_m)\Big| = \big|L(f(e^{it}\cdot)\big| \leq \|L\| \cdot \|f\|.$$

This allows us to use the Lebesgue domination convergence theorem to guarantee that $L(f(e^{it}\cdot))$ defines an element in $L_1([0, 2\pi])$. The fact that L is τ_0-continuous implies that there is compact subset M of B_H, which we can suppose to be balanced, and $A > 0$, such that

$$|L(f)| \leq A \sup_{z \in M} |f(z)|.$$

This leads to $|L(f(e^{it}\cdot)| \leq A \sup_{z \in M} |f(z)|$. So the linear map $f \in X \rightsquigarrow L(f(e^{it}\cdot)) \in L_1([0, 2\pi])$ is τ_0-continuous.

Further the linear functional $\Lambda : L_1([0, 2\pi]) \to \mathbb{C}$ given by $\Lambda(h) = \frac{1}{2\pi} \int_0^{2\pi} \frac{h(t)}{e^{it}} dt$ is a continuous one, hence the linear functional, F, on X given by

$$f \in X \overset{F}{\rightsquigarrow} \Lambda(L(f(e^{it}\cdot))) = \frac{1}{2\pi} \int_0^{2\pi} \frac{L(f(e^{it}\cdot))}{e^{it}} dt$$

is τ_0-continuous. This together with the fact that $\sum_{m=0}^{\infty} e^{imt} f_m$ is τ_0-convergent to $f(e^{it}\cdot)$, leads to

$$F(f) = \frac{1}{2\pi} \int_0^{2\pi} \frac{L(f(e^{it}\cdot))}{e^{it}} dt = \frac{1}{2\pi} \sum_{m=0}^{\infty} \int_0^{2\pi} e^{imt} \frac{L(f_m)}{e^{it}} dt = \frac{1}{2\pi} \sum_{m=0}^{\infty} L(f_m) \int_0^{2\pi} e^{i(m-1)t} dt = L(f_1).$$

Since $f_1(z) =< z, \overline{\nabla f(0)} >= \nabla f(0)^*(z)$, we conclude that $F(f) = L(\nabla f(0)^*)$, and that there is a constant $C > 0$ such that

$$|L(\nabla f(0)^*)| \leq C \|f\|. \tag{4}$$

Now, we fix an orthonormal basis $\{e_j\}_{j \in J}$ in H, and we claim that $\sum L(e_j^*) e_j$ defines an element in H. For $(\alpha_j) \in H$, the net of partial sums $\big(\sum_{j \in \gamma} \alpha_j e_j\big)_{\gamma \in \Gamma}$ (Γ the ordered set of finite subsets of J) is known to converge in H to (α_j). This leads to $\lim_\gamma \sum_{j \in \gamma} \alpha_j (e_j^*)(z) = \lim_\gamma \big(\sum_{j \in \gamma} \alpha_j (e_j^*)\big)(z) = (\alpha_j)^*(z)$ uniformly on B_H, hence $\lim_\gamma \sum_{j \in \gamma} \alpha_j e_j^* = (\alpha_j)^*$ also in the compact open topology, so $\lim_\gamma \sum_{j \in \gamma} \alpha_j L(e_j^*) = L((\alpha_j)^*)$. Further,

$$\Big|\sum_{j \in \gamma} \alpha_j L(e_j^*)\Big| = \Big|L(\sum_{j \in \gamma} \alpha_j e_j^*)\Big| \leq A \sup_{z \in M} \Big|\sum_{j \in \gamma} \alpha_j e_j^*(z)\Big| = A \sup_{z \in M} \Big|\sum_{j \in \gamma} \alpha_j z_j\Big| \leq A \Big\|(\alpha_j)\Big\|.$$

Thus by the uniform boundedness principle, the linear form $(\alpha_j) \in H \rightsquigarrow \sum \alpha_j L(e_j^*)$ is a continuous one. That is $\varpi = (\overline{L(e_j^*)}) \in H$ and $\varpi \neq 0$ since $L(e_1^*) \neq 0$, and $L((\alpha_j)^*) = \sum \alpha_j L(e_j^*) =< (\alpha_j), \varpi >$. Put $\upsilon = \frac{\varpi}{\|\varpi\|}$.

Next, for any $f \in X$ with $\nabla f(0) \neq 0$, we may find an isometric isomorphism ϕ, hence an automorphism of the ball exchanging υ and $\frac{\nabla f(0)}{\|\nabla f(0)\|}$. Then for $g := f \circ \phi$, we have $g'(0) = f'(0) \circ \phi$, so $< \nabla g(0), z >=< \nabla f(0), \phi(z) >$. Therefore,

$$|L(\nabla g(0)^*)| = | < \nabla g(0), \varpi > | = \|\varpi\| | < \nabla g(0), \upsilon > | =$$

$$\|\varpi\| | < \nabla f(0), \phi(\upsilon) > | = \|\varpi\| < \nabla f(0), \frac{\nabla f(0)}{\|\nabla f(0)\|} > | = \|\varpi\| \|\nabla f(0)\|.$$

Now inequality (4) yields

$$\|\varpi\| \|\nabla f(0)\| \leq C\|g\| = C\|f\| \text{ that is, } \|\nabla f(0)\| \leq \frac{C}{\|\varpi\|}\|f\|.$$

An inequality also valid if $\nabla f(0) = 0$. And now for the invariant gradient,

$$\|\tilde{\nabla} f(z)\| = \|\nabla(f \circ \varphi_z)(0)\| \leq \frac{C}{\|\varpi\|}\|f \circ \varphi_z\| = \frac{C}{\|\varpi\|}\|f\|$$

which shows that $f \in \mathcal{B}(B_H)$.

Assume now that $L(1) \neq 0$. Instead of $\Lambda : L_1([0, 2\pi]) \to \mathbb{C}$ we consider the linear functional Ω given by $\Omega(h) = \frac{1}{2\pi} \int_0^{2\pi} h(t)dt$ and argue analogously. Then, the linear functional, G, on X given by

$$f \in X \overset{G}{\rightsquigarrow} \Omega(L(f(e^{it} \cdot))) = \frac{1}{2\pi} \int_0^{2\pi} L(f(e^{it} \cdot))dt$$

is τ_0-continuous and $G(f) = L(f_0) = f(0)L(1)$. In addition, there is a constant $B > 0$ such that $|f(0)L(1)| = |G(f)| \leq B\|f\|$. Now, replacing f by $f \circ \varphi_z$, we get $|f(z)L(1)| \leq B\|f \circ \varphi_z\| = \|f\|$. That is, f is bounded and $\|f\|_\infty \leq \frac{B}{|L(1)|}\|f\|$. □

3. MULTIPLIERS

Recall that a function f is said to be a multiplier for the Bloch space if $fg \in \mathcal{B}(B_H)$ for all $g \in \mathcal{B}(B_H)$.

The key to characterize the multipliers for the Bloch space in the n-ball $\mathbb{B}_n$ is the following result that for $x, y \in \mathbb{B}_n$ we have

$$\beta(x, y) = \sup \{|f(x) - f(y)| : ||f||_{\mathcal{B}} \leq 1\},$$

where β is the Bergman or hyperbolic metric in $\mathbb{B}_n$ and $f : \mathbb{B}_n \to \mathbb{C}$ is an analytic function on $\mathbb{B}_n$.

The same result was established for arbitrary Hilbert spaces H in [3, Corollary 3.5]. And accordingly, the characterization of the multipliers for $\mathcal{B}(B_H)$ follows in the very same way as in the finite dimensional case, see [9, Theorem 3.21].

Theorem 3.1. *Let $f \in \mathcal{H}(B_H)$. Then f is a multiplier of the Bloch space $\mathcal{B}(B_H)$ if and only if $f \in H^\infty(B_H)$ and the function $z \in B_H \rightsquigarrow (1-\|z\|^2)\|\nabla f(z)\| \log \frac{1}{1-\|z\|^2}$ is bounded.*

Proof. If f is a multiplier of the Bloch space, then the closed graph theorem shows that there is a constant $C > 0$ such that $\|fg\| \leq C\|g\|$ for all $g \in \mathcal{B}(B_H)$. To check that $f \in H^\infty(B_H)$ it suffices to realize that

$$|f(z)||\delta_z(g)| = |f(z)g(z)| = |\delta_z(fg)| \leq \|fg\|\|\delta_z\| \leq \|\delta_z\|C\|g\|,$$

and taking supremum for $\|g\| \leq 1$, we get $|f(z)|\|\delta_z\| \leq \|\delta_z\|C$, thus $|f(z)| \leq C$. That is, $f \in H^\infty(B_H)$.

Since $\nabla(fg)(z) = f(z)\nabla g(z) + g(z)\nabla f(z)$, we get

$$|g(z)|\|\nabla f(z)\|(1-\|z\|^2) \leq \|f\|_\infty\|g\| + C\|g\| \quad \text{for all } g \in \mathcal{B}(B_H) \text{ and all } z \in B_H. \tag{5}$$

As mentioned above, for $x, y \in B_H$ we have

$$\beta_H(x,y) = \sup\left\{|g(x) - g(y)| : ||g||_{inv} \leq 1\right\},$$

where β denotes the hyperbolic distance in B_H. So by taking supremum on g in the unit ball of $\mathcal{B}(B_H)$ and $g(0) = 0$, we obtain that $(1-\|z\|^2)\|\nabla f(z)\| \log \frac{1}{1-\|z\|^2}$ is bounded.

We omit the proof of the reverse condition as it mimics the one for $\mathcal{B}(\mathbb{B}_n)$. □

By $\mathcal{B}_0(B_H)$ we denote *the little Bloch* space

$$\{f \in \mathcal{B}(B_H) : \lim_{\|x\|\to 1^-}(1-\|x\|^2)|\mathcal{R}f(x)| = 0\}$$

as defined in [4]. Recall that $\mathcal{R}f(x) :=< x, \overline{\nabla f(x)} >$ is the radial derivative of f at x.

The growth of a function in $\mathcal{B}_0(B_H)$ behaves in the same way as in the finite dimensional case.

Lemma 3.1. *If $g \in \mathcal{B}_0(B_H)$, then* $\lim_{\|x\|\to 1^-} \frac{g(x)}{\log \frac{1}{1-\|x\|^2}} = 0$.

Proof. We can assume WLOG that $g(0) = 0$ and $\|g\| = 1$. Let $\epsilon > 0$. Then there is $\frac{1}{2} < s < 1$ such that $(1-\|y\|^2)|\mathcal{R}g(y)| \leq \epsilon$ if $\|y\| > s^2$.

So,

$$|g(sx)| = |\delta_{sx}(g)| \leq \log\frac{1+\|sx\|}{1-\|sx\|} \leq \log\frac{1+s}{1-s}.$$

Choose now $r > s$ such that for $\|x\| > r$, we have $\log\frac{1}{1-\|x\|^2} > \frac{\log\frac{1+s}{1-s}}{\epsilon}$ and hence $\frac{|g(sx)|}{\log\frac{1}{1-\|x\|^2}} \leq \epsilon$.

Moreover if $\|x\| > s$,

$$\begin{aligned} |g(x) - g(sx)| &= \left|\int_s^1 g'(xt)(x)dt\right| = \left|\int_s^1 \frac{1}{t}\mathcal{R}g(xt)dt\right| = \left|\int_s^1 \frac{1}{t}\frac{\mathcal{R}g(xt)(1-\|xt\|^2)}{1-\|xt\|^2}dt\right| \\ &\leq \frac{\epsilon}{\|x\|^2}\int_s^1 \frac{\|x\|}{1-\|x\|^2|t|^2}dt \leq 2\epsilon\log\frac{1+\|x\|}{1-\|x\|}. \end{aligned}$$

Since $\log \frac{1+\|x\|}{1-\|x\|} = \mathcal{O}\Big(\log \frac{1}{1-\|x\|^2} \Big)$ when $\|x\| \to 1$, it follows that

$$\Big| \frac{g(x)}{\log \frac{1}{1-\|x\|^2}} \Big| \leq \frac{|g(sx)|}{\log \frac{1}{1-\|x\|^2}} + K\epsilon \leq \epsilon(1+K) \ \text{ if } \|x\| > r.$$

□

Corollary 3.1. *The function f is a multiplier of the Bloch space $\mathcal{B}(B_H)$ if and only if f is a multiplier of the little Bloch space $\mathcal{B}_0(B_H)$.*

Proof. Let $g \in \mathcal{B}_0(B_H)$. Suppose that f is a multiplier of $\mathcal{B}(B_H)$. Since $\lim_{\|x\|\to 1^-} (1-\|x\|^2)|\mathcal{R}g(x)| = 0$, also

$$\lim_{\|x\|\to 1^-} (1-\|x\|^2)|f(x)||\mathcal{R}g(x)| = 0.$$

On the other hand,

$$(1-\|x\|^2)|g(x)||\mathcal{R}f(x)| \leq \frac{|g(x)|}{\log \frac{1}{1-\|x\|^2}} (1-\|x\|^2)\|\nabla f(x)\| \log \frac{1}{1-\|x\|^2}.$$

Hence using Lemma 3.1 and Theorem 3.1, we get that $\lim_{\|x\|\to 1^-} g(x)(1-\|x\|^2)\mathcal{R}(f)(x) = 0$. And since $\mathcal{R}(fg)(x) = g(x)\mathcal{R}f(x) + f(x)\mathcal{R}g(x)$, we deduce that

$$\lim_{\|x\|\to 1^-} (1-\|x\|^2)\mathcal{R}(fg)(x) = 0.$$

Thus $fg \in \mathcal{B}_0(B_H)$.

For the converse, suppose that f is a multiplier of $\mathcal{B}_0(B_H)$. Then there is $C > 0$ such that $\|fh\| \leq C\|h\|$ for all $h \in \mathcal{B}_0(B_H)$. Let $g \in \mathcal{B}(B_H)$. Using [3, Theorem 3.1] it suffices to prove that

$$\sup \left\{ \frac{\big|(fg)(x) - (fg)(y)\big|}{\beta_H(x,y)} : x, y \in B_H,\ x \neq y \right\} < \infty.$$

Consider for $0 < r < 1$ the functions $g_r(x) := g(rx)$, which belong to $\mathcal{B}_0(B_H)$. Thus $fg_r \in \mathcal{B}_0(B_H)$ by assumption and, moreover, $\|g_r\| \leq \|g\|$, hence $\|fg_r\| \leq C\|g_r\| \leq C\|g\|$. Appealing again to [3, Theorem 3.1] and the equivalence of the semi-norms $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{inv}$, there is a constant $A > 0$ such that

$$\sup \left\{ \frac{\big|(fg_r)(x) - (fg_r)(y)\big|}{\beta_H(x,y)} : x, y \in B_H,\ x \neq y \right\} \leq A.$$

Letting $r \to 1^-$, we obtain

$$\sup \left\{ \frac{\big|(fg)(x) - (fg)(y)\big|}{\beta_H(x,y)} : x, y \in B_H,\ x \neq y \right\} \leq A,$$

as needed. □

Remark 3.1. *The vector space $\mathcal{B}_0(B_H) \bigcap H^\infty(B_H)$ is a Banach subalgebra of $H^\infty(B_H)$.*

Proof. If $f, g \in \mathcal{B}_0(B_H) \bigcap H^\infty(B_H)$, then it follows in an easier way than in the above corollary that $fg \in \mathcal{B}_0(B_H) \bigcap H^\infty(B_H)$.

Any $\|\cdot\|_\infty$-Cauchy sequence in $\mathcal{B}_0(B_H) \bigcap H^\infty(B_H)$ is also a Cauchy sequence in $\mathcal{B}(B_H)$. Hence its limit belongs to both $\mathcal{B}_0(B_H)$ and $H^\infty(B_H)$. □

Lemma 3.2. *The multiplication operator $M_f : \mathcal{B}(B_H) \to \mathcal{B}(B_H)$ given by $M_f(g) = gf$, is invertible if and only if $\frac{1}{f} \in H^\infty(B_H)$.*

Proof. If M_f is invertible, there is $h \in \mathcal{B}(B_H)$ such that $fh = 1$. Thus, $f(x) \neq 0$ for all $x \in B_H$ and so, $\frac{1}{f} \in \mathcal{H}(B_H)$. Further, $\frac{1}{f}$ is a multiplier for $\mathcal{B}(B_H)$ since for each $g \in \mathcal{B}(B_H)$, there is $h \in \mathcal{B}(B_H)$ such that $fh = M_f(h) = g$, hence $M_{\frac{1}{f}} g = \frac{1}{f} g = h \in \mathcal{B}(B_H)$. Now, apply Theorem 3.1.

If $\frac{1}{f} \in H^\infty(B_H)$, then there is $a > 0$ such that $a \leq |f(x)|$ for all $x \in B_H$. In order to prove that M_f is invertible, it suffices to check that $\frac{1}{f}$ is a multiplier for $\mathcal{B}(B_H)$. That is, to verify that $\frac{1}{f}$ satisfies the condition in Theorem 3.1. Indeed:

Since $\nabla \frac{1}{f}(x) = \frac{-1}{f^2(x)} \nabla f(x)$, we have

$$\left\|\nabla\left(\frac{1}{f}\right)(x)\right\| = \left|\frac{-1}{f^2(x)}\right| \left\|\nabla f(x)\right\| \leq \frac{1}{a^2} \left\|\nabla f(x)\right\|$$

which together with the fact that f fulfills the condition in Theorem 3.1 yields the result. □

Theorem 3.2. *Assume* $\dim(H) > 1$. *The spectrum $\sigma(M_f)$ and the essential spectrum $\sigma_e(M_f)$ of the multiplication operator $M_f : \mathcal{B}(B_H) \to \mathcal{B}(B_H)$ coincide with $\overline{f(B_H)}$. Further $\sigma_e(M_f) = \bigcap_{0<r<1} \overline{f(B_H \setminus rB_H)} = \sigma(M_f)$.*

Proof. Notice that $M_f - \lambda Id = M_{f-\lambda}$. By Lemma 3.2, $M_{f-\lambda}$ is invertible if and only if, $f - \lambda$ is bounded below, which is equivalent to $\lambda \notin \overline{f(B_H)}$.

For the essential spectrum, we show that $f(B_H) \subset \sigma_e(M_f)$. First, we notice that the set of evaluations at points in B_H is linearly independent in $\mathcal{B}(B_H)^*$: Indeed, if $\sum_{j=1}^m \alpha_j \delta_{x_j} = 0$ and because every finite subset of B_H is linear interpolating for $H^\infty(B_H)$, we may find $F_j \in H^\infty(B_H) \subset \mathcal{B}(B_H)$, such that $F_l(x_j) = \delta_j^l$, thus $0 = \left(\sum_{j=1}^m \alpha_j \delta_{x_j}\right) F_l = \alpha_l$.

Fix $\lambda \in f(B_H)$. We may assume $f \neq 0$. Since f has no isolated zeroes, there is an infinite number of them, say $\{x_j\}$. It turns out that all $\delta_{x_j} \in Ker M_{f-\lambda}^*$, the adjoint map of $M_{f-\lambda}$. Hence $M_{f-\lambda}^*$ is not a Fredholm operator, so neither is $M_{f-\lambda}$. Therefore, $\lambda \in \sigma_e(M_f)$, as wanted. To conclude, recall that the essential spectrum is a closed subset of the spectrum.

For the second statement, let $\lambda \notin \bigcap_{0<r<1} \overline{f(B_H \setminus rB_H)}$. Then there are $r \in (0,1)$ and $\delta > 0$ such that $|\lambda - f(x)| \geq \delta$ for all $r \leq \|x\| < 1$. Then $g(x) = (f(x) - \lambda)^{-1}$ is analytic and bounded on $B_H \setminus r\overline{B}_H$. By Hartogs' extension type theorem from [5, Theorem 5] extend g to $\tilde{g}$ analytic on B_H such that $\tilde{g}(x) = (f(x) - \lambda)^{-1}$ for all $x \in B_H \setminus r\overline{B}_H$. Notice that if g is bounded, then Hartogs' extension $\tilde{g}$ is also bounded because for the restriction $\tilde{g}_{|_{r\overline{B}_H}}$ and $x \in r\overline{B}_H$, we have $|\tilde{g}(x)| \leq \sup_{\|u\|=r} |\tilde{g}(u)| \leq \frac{1}{\delta}$ thanks to the maximum norm theorem (see [1, Proposition 10.2]). Clearly $h(x) := \tilde{g}(x)(f(x) - \lambda) \in \mathcal{H}(B_H)$ and $h(x) = 1$ if $x \in B_H \setminus r\overline{B}_H$.

Now the identity principle [8, Proposition 5.7], gives that $\tilde{g}(x) = (f(x) - \lambda)^{-1}$ for all $x \in B_H$ and $(f - \lambda)^{-1} \in H^\infty(B_H)$. Hence $M_{f-\lambda}$ is invertible by Lemma 3.2, so $\lambda \notin \sigma(M_f)$.

□

We are able to extend [6, Corollary 1] to our arbitrary dimensional setting. Indeed, from Theorem 3.2 we conclude directly that M_f acting on $\mathcal{B}(B_H)$ is not compact unless $f = 0$.

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